QUASI-COMPACTNESS AND ABSOLUTELY CONTINUOUS KERNELS APPLICATIONS TO MARKOV CHAINS

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Abstract : We show how the essential spectral radius $r_e(Q)$ of a bounded positive kernel Q, acting on bounded functions, is linked to the lower approximation of Q by certain absolutely continuous kernels. The standart Doeblin's condition can be interpreted in this context, and, when suitably reformulated, it leads to a formula for $r_e(Q)$. This results may be used to characterize the Markov kernels having a quasi-compact action on a space of measurable functions bounded with respect to some test function, when no irreducibilty and aperiodicity are assumed.

I. INTRODUCTION

Let $(E, \mathcal{E})$ be a measurable space.

Definition I.1

A function Q from $E \times \mathcal{E}$ to $\mathbb{R}_+$ is a bounded positive kernel if

(i) $\forall A \in \mathcal{E}$, $Q(\cdot, A)$ *is $\mathcal{E}$-measurable,*

(ii) $\forall x \in E$, $Q(x, \cdot)$ *is a positive measure on* $(E, \mathcal{E})$,

(iii) $\sup_{x \in E} Q(x, E) < +\infty$.

We shall denote by $\mathcal{T}(E, \mathcal{E})$ the cone of bounded positive kernels on $(E, \mathcal{E})$.

Set, for any positive measurable f and $x \in E$,

$$Qf(x) = \int_E f(y) Q(x, dy).$$

Then the kernel Q defines a bounded operator on the Banach space $\mathcal{B}$ of bounded measurable complex valued functions on $(E, \mathcal{E})$ equipped with the supremum norm. The aim of the paper is to state conditions for the quasi-compactness and to give a formula for the essential spectral radius of kernels $Q \in \mathcal{T}(E, \mathcal{E})$ acting on $\mathcal{B}$. In fact, we shall partially extend the domain of our study to the family $(Q_\chi)_{\chi \in \mathcal{X}}$ of bounded operators on $\mathcal{B}$ associated with Q and indexed by the elements χ of the space $\mathcal{X}$ of bounded measurable complex valued functions on $E \times E$; the kernels Q_χ are defined by

$$Q_\chi f(x) = \int_E f(y) \chi(x, y) Q(x, dy).$$

Let w be a measurable function from $(E, \mathcal{E})$ to $[1, +\infty[$. The kernels Q and Q_χ may also act on the space $\mathcal{B}_w$ of complex valued measurable functions f on $(E, \mathcal{E})$ verifying $\sup_E w^{-1}|f| < +\infty$, endowed with the norm $\|f\|_w = \sup_E w^{-1}|f|$. So one may ask how to estimate the essential spectral radius of Q and Q_χ in this context. It appears that an answer can be given by the use of a conjugate kernel acting on $\mathcal{B}$. If Q is a Markov kernel, its conjugate kernel is no more Markov ; this is one reason to study bounded positive kernels.

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Let us point out the usefulness of the quasi-compactness properties for a Markov kernel P.

First quasi-compactness on $\mathcal{B}_w$ allows to describe the asymptotic behaviour of the sequence of iterated powers $(P^n)_{n\geq 0}$ in terms of a strong ergodic theorem, or even of a uniform ergodic theorem in the case where 1 is the only eigenvalue of modulus 1 of P and is simple (uniform geometric ergodicity). We refer to [Nev], [BR], [Rev], and the early Yosida Kakutani's Ergodic Theorem (1941), [DS] VIII.8.6, for the case of a quasi-compact action on $\mathcal{B}_1 = \mathcal{B}$. The general case is treated in Corollary IV.3.

Secondly, let ξ be a measurable real valued function on E. Following Nagaev [Nag], several works, see [HenHer] for an overview, have shown how a property of quasi-compactness of P and of the Fourier kernels associated with P and ξ can be used to obtain limit theorems for the sequence of real random variables $\big(\xi(X_n)\big)_{n\geq 0}$. In the present setting, the Fourier kernel $P(t)$, $t \in \mathbb{R}$, is P_{χ_t}, with $\chi_t(x,y) = e^{it\xi(y)}$, $x, y \in E$. Suppose moreover that for some $s \neq 0$, $\sup_{x\in E} \int_E e^{s\xi(y)} P(x,dy) < +\infty$, then the Fourier-Laplace kernel $\widetilde{P}(s+it)$ is Q_{χ_t}, with $Q(x,dy) = e^{s\xi(y)}P(x,dy)$ and with χ_t as above. These kernels give a tool for the study of large deviations of the sequence $(\xi(X_n))_{n\geq 0}$. The case of Laplace kernels gives a second reason for the study of positive bounded kernels rather than Markov kernels, even for applications to Markov chains. Finally, recall that quasi-compactness is also useful to describe the stochastic behaviour of a dynamical system, when a Perron-Frobenius operator can be associated with the given measure preserving transformation. Indeed, this can be viewed as a Markov chain behaviour [HenHer].

Our main results are stated and proved in Section III. We show how the essential spectral radius of a kernel Q acting on $\mathcal{B}$ is linked to the lower approximation of Q by elements of a class $\mathcal{K}^*$ of bounded positive absolutely continuous kernels. An absolutely continuous kernel is a kernel which is defined by means of a probability measure and of a measurable function on $E \times E$; in order to belong to $\mathcal{K}^*$ such a kernel has to satisfy a condition of uniform integrability which appears to be equivalent to the weak compactnesss of its action on the space of bounded complex measures. Using differentiation of measures, we express the preceding results in terms of generalized Doeblin's conditions.

We then consider the case of Markov kernels, Section IV. With the help of the previous study, we characterize the general Markov kernels P having a quasi-compact action on a space $\mathcal{B}_w$. This leads to generalize a result only known for irreducible and aperiodic kernels.

The key tool in Section III is a Nussbaum's formula for the essential spectral radius. It is recalled in Section II, together with some results on quasi-compactness.

Doeblin's work is of course the first one on the subject. Among its improvements mention the paper of R. Fortet [For]. More recently L. Wu [Wu] has obtained bounds for the essential spectral radius, see Remark III.2.

II. ESSENTIAL SPECTRAL RADIUS, NUSSBAUM's FORMULA

In this section $\mathcal{B}$ is an abstract Banach space, $\mathcal{L}(\mathcal{B})$ is the Banach algebra of bounded operators on $\mathcal{B}$, and $Q \in \mathcal{L}(\mathcal{B})$. We denote by $r(Q)$ the spectral radius of Q, and by $Q_{|G}$ its restriction to a Q-invariant subspace G. The essential spectral radius of Q may be defined as follows.

Definition II.1

The essential spectral radius of $Q \in \mathcal{L}(\mathcal{B})$, denoted by $r_e(Q)$, is the infimum of $r(Q)$ and of the real numbers $\rho \geq 0$ such that we have

$$\mathcal{B} = F_\rho \oplus H_\rho,$$

where F_ρ and H_ρ are Q-invariant subspaces such that H_ρ is closed and $r(Q_{|H_\rho}) < \rho$, $\dim F_\rho < +\infty$ and the eigenvalues of $Q_{|F_\rho}$ have a modulus $\geq \rho$.

When $r_e(Q) < r(Q)$, the operator Q is said to be quasi-compact.

Assume that Q is a quasi-compact operator on $\mathcal{L}(\mathcal{B})$ and let $r_e(Q) < \rho < r(Q)$. If Π is the projector onto F_ρ in the above direct sum decomposition, the Closed Graph Theorem implies that Π is a bounded operator. Setting $L = Q\Pi$ and $N = Q(I - \Pi)$, we have, for any $n \geq 0$,

$$Q^n = L^n + N^n.$$

It follows that

$$\lim_n \rho^{-n} \|Q^n - L^n\| = 0.$$

So, at order $(\rho^n)_{n\geq 0}$, the asymptotic behaviour of the iterated powers Q^n, $n \geq 0$, is described by the iterated powers of the finite rank operator L.

R. D. Nussbaum [Nus] has established two formulas for the essential spectral radius of an operator. One of these is based on the use of a set function γ which measures the non compactness of subsets in $\mathcal{B}$. Nussbaum shows how $r_e(Q)$ is linked to the way the iterated powers Q^n, $n \geq 1$, act on γ. This formula has been successfully used [Hen1] to weaken the hypotheses and to get an upper bound for the essential spectral radius in the Theorem of Ionescu Tulcea Marinescu [ITM]. The other Nussbaum's formula is based on approximation by compact operators. It appears to be convenient to the present study. Let us recall this formula.

Theorem II.1

Let $\mathcal{K}(\mathcal{B})$ be the ideal of compact operators on $\mathcal{B}$. For any $Q \in \mathcal{L}(\mathcal{B})$, we have

$$r_e(Q) = \lim_n \big(\inf\{\|Q^n - V\| : V \in \mathcal{K}(\mathcal{B})\}\big)^{1/n}.$$

In the course of our study, we shall need the properties of $r_e(\cdot)$ collected in the following statement.

Corollary II.1

Let $Q \in \mathcal{L}(\mathcal{B})$. Then

(i) for $n \geq 1$, $r_e(Q) = \big(r_e(Q^n)\big)^{1/n}$,

(ii) if G is a Q-invariant closed subspace of $\mathcal{B}$, then $r_e(Q_{|G}) \leq r_e(Q)$,

(iii) let $\mathcal{B}'$ be the topological dual space of $\mathcal{B}$ and let Q' be the adjoint of Q, then

$$r_e(Q') = r_e(Q).$$

To be complete, main elements of the proofs of the results stated above are given in Section V.

To end this section, we prove a lemma which happens to be useful when dealing with quasi-compactness of operators belonging to a closed subalgebra of $\mathcal{L}(\mathcal{B})$.

Lemma II.2

Let $\mathcal{A}$ be a closed subalgebra of $\mathcal{L}(\mathcal{B})$. Assume that $Q \in \mathcal{A}$ is quasi-compact.

Then, for any ρ, $r_e(Q) < \rho < r(Q)$, the projector Π on F_ρ associated with the direct sum decomposition $\mathcal{B} = F_\rho \oplus H_\rho$ of Definition II.1 belongs to $\mathcal{A}$. Consequently, there exist $L \in \mathcal{A}$ and $N \in \mathcal{A}$ such that

$$Q = L + N, \qquad LN = NL = 0,$$

$r(N) < \rho$, L has a finite rank, and its non zero eigenvalues have a modulus $\geq \rho$.

Proof

As seen in the lines following Definition II.1, the operators $L = Q\Pi$ and $N = (I-\Pi)Q$ verify the stated properties, so that we have only to show that $\Pi \in \mathcal{A}$.

Denote by $\sigma(Q)$ the spectrum of Q, and let $R(z) = (z-Q)^{-1}$ be the resolvent of Q at $z \notin \sigma(Q)$. Clearly $\sigma(Q) = \sigma(Q_{|F_\rho}) \cup \sigma(Q_{|H_\rho})$. Choose ρ_0, ρ_1, ρ_2, such that $r(Q_{|H_\rho}) < \rho_0 < \rho_1 < \rho \leq r(Q_{|F_\rho}) < \rho_2$. Let Γ be the positively oriented boundary of the ring $\{z : z \in \mathbb{C}, \rho_1 < |z| < \rho_2\}$, then we have,

$$\Pi = \frac{1}{2i\pi} \int_\Gamma R(z) dz,$$

see [DS] VII.3, or [Hen2] where an elementary proof adapted to quasi-compactness is given. As $\mathcal{A}$ is closed, to prove that $\Pi \in \mathcal{A}$, it suffices to show that, for $z \in \Gamma$, $R(z) \in \mathcal{A}$.

Set $U = \{z : z \in \mathbb{C}, \rho_0 < |z|, z \notin \sigma(Q_{|F_\rho})\}$. Since $\sigma(Q_{|F_\rho})$ is finite, U is a connected open subset of $\mathbb{C}$. Let $\Omega = \{z : z \in U,\ R(z) \in \mathcal{A}\}$. Using the fact that $\mathcal{A}$ is a Banach algebra, it is easily verified that Ω is non empty and open ; moreover the continuity of $R(\cdot)$ implies that Ω is closed. Since U is connected, we conclude that $\Omega = U$, this achieves the proof. []

III. POSITIVE KERNELS ACTING ON $\mathcal{B}$

In this section, we first establish an upper bound for the essential spectral radii of the kernels Q_χ acting on the Banach space $\mathcal{B}$ of bounded measurable complex valued functions on $(E, \mathcal{E})$, endowed with the supremum norm $||\cdot||$, Theorem III.1. As far as Q is concerned, Theorem III.1 has a converse, Theorem III.2, giving a lower bound. Collecting these two results, we get an exact formula for the essential spectral radius of Q, Theorem III.3.

Notice that the hypotheses of Theorem III.1 involve an upper bound on $r(S)$, while in the assertions of Theorem III.2 the corresponding bound is on $\|S\|$. So Theorem III.1 may appear needlessly general, this is invalidate by the applications stated in Section IV.

Let $Q \in \mathcal{T}(E, \mathcal{E})$ and $\chi \in \mathcal{X}$. We shall denote by $r(Q_\chi)$ the spectral radius of Q_χ acting on $(\mathcal{B}, \|\cdot\|)$, and by $r_e(Q_\chi)$ the essential spectral radius of Q_χ on the same space. It is easily seen that $r(Q) = \lim_n \|Q^n 1_E\|^{1/n}$. We equip the space of parameter $\mathcal{X}$ with the norm $\|\chi\| = \sup_{E\times E} |\chi|$.

III.1 Upper bounds

We introduce the kind of positive absolutely continuous kernels which is at the center of our study.

Definition III.1

We denote by $\mathcal{P}$ the set of probability measure on $(E, \mathcal{E})$.

For $\nu \in \mathcal{P}$, $\mathcal{H}_\nu$ is the set of positive measurable functions α on $(E \times E, \mathcal{E} \otimes \mathcal{E})$ such that the functions $\alpha(x, \cdot)$, $x \in E$, are uniformly ν-integrable, that is

$$\lim_{m\to+\infty} \sup_{x\in E} \int_{\{y:\alpha(x,y)\geq m\}} \alpha(x,y)d\nu(y) = 0.$$

With each $\alpha \in \mathcal{H}_\nu$, we associate the bounded positive kernel $T_{\nu,\alpha}$ defined by

$$T_{\nu,\alpha}(x, A) = \int_A \alpha(x, y)d\nu(y), \quad (x, A) \in E \times \mathcal{E}.$$

Theorem III.1

Let $Q \in \mathcal{T}(E, \mathcal{E})$. Assume that there exist an integer $\ell \geq 1$, $\nu \in \mathcal{P}$, and $\alpha \in \mathcal{H}_\nu$ such that $S = Q^\ell - T_{\nu,\alpha} \geq 0$ and $r(S)^{1/\ell} < r(Q)$.

Then

(i) the operator Q is quasi-compact and $r_e(Q) \leq r(S)^{1/\ell}$,

(ii) assume moreover that either $\ell = 1$, or $\mathcal{E}$ is countably generated, then, if $\chi \in \mathcal{X}$ is such that $\|\chi\| \, r(S)^{1/\ell} < r(Q_\chi)$, the operator Q_χ is quasi-compact and $r_e(Q_\chi) \leq \|\chi\| \, r(S)^{1/\ell}$.

The inequality $Q^\ell - T_{\nu,\alpha} \geq 0$ means that, for each $(x, A) \in E \times \mathcal{E}$, we have $Q^\ell(x, A) \geq T_{\nu,\alpha}(x, A)$. To establish a link between the assertions *(i)* and *(ii)* it must be noticed that

$$r(Q_\chi) \leq \|\chi\| \, r(Q).$$

As mentioned in the introduction, the case $\|\chi\| \leq 1$ is of particular interest, it motivates the following obvious consequence of the above theorem.

Corollary III.1

Let Q, ℓ, ν, α, S and $\mathcal{E}$ be as in Theorem III.1.

Then, for any $\chi \in \mathcal{X}$ such that $\|\chi\| \leq 1$, we have either $r(Q_\chi) = r(Q)$ and Q_χ is quasi-compact, or $r(Q_\chi) < r(Q)$.

Proof of Theorem III.1

As quasi-compactness of Q_χ means $r_e(Q_\chi) < r(Q_\chi)$, the assertions of the theorem may be briefly stated $r_e(Q_\chi) \leq \|\chi\| \, r(S)^{1/\ell}$. We shall establish this inequality using the Nussbaum's formula recalled in the previous section.

We denote by $\mathcal{I}_\nu$ the set of complex valued measurable functions a on $E \times E$ such that

$$\sup_{x\in E} \int |a(x, y)|d\nu(y) < +\infty,$$

and by $\mathcal{H}_\nu^{\mathbb{C}}$ the subset of functions a such that the functions $a(x, \cdot)$, $x \in E$, are uniformly ν-integrable.

As was done for a function in $\mathcal{H}_\nu$, we associate with $a \in \mathcal{I}_\nu$ the bounded linear operator $T_{\nu,a}$ on $\mathcal{B}$ defined by $T_{\nu,a}f(x) = \int f(y)a(x, y)d\nu(y)$. Since in this proof, ν is fixed, we shall use the short notation $T_a = T_{\nu,a}$. The key result about the kernels T_a is the following.

Proposition III.1

Let $a \in \mathcal{H}_\nu^{\mathbb{C}}$.

(a) If $a' \in \mathcal{I}_\nu$, Then $T_aT_{a'}$ is a compact operator of $(\mathcal{B}, \|\cdot\|)$.

(b) Let $S \in \mathcal{T}(E, \mathcal{E})$, $\chi \in \mathcal{X}$, and $k \geq 1$, we have $(S_\chi)^kT_a = T_{a_k}$ with $a_k \in \mathcal{H}_\nu^{\mathbb{C}}$.

Proof of Proposition III.1

Proof of (a)

Each $f \in \mathcal{B}$ is ν-integrable, hence setting $||f||_1 = \int_E |f| d\nu$, we get a semi-norm on $\mathcal{B}$ which verifies $||f||_1 \le ||f||$. For $f \in \mathcal{B}$ and $r > 0$, we set

$$B(f,r) = \{g : g \in \mathcal{B}, \|g - f\| < r\}, \qquad B_1(f,r) = \{g : g \in \mathcal{B}, \|g - f\|_1 < r\}.$$

A subset C of $\mathcal{B}$ will be said to be $\|\cdot\|$-totally bounded (resp. $\|\cdot\|_1$-totally bounded) if, for any $\epsilon > 0$, there exists a finite covering of C by ball of type $B(f,r)$ (resp. $B_1(f,r)$).

Lemma III.1

Let $a \in \mathcal{I}_\nu$.

(i) If $U = \{f : f \in \mathcal{B}, \|f\| \le 1\}$, then $T_a(U)$ is $\|\cdot\|_1$-totally bounded,

(ii) if $\sup_{E\times E} |a| = m < +\infty$, then, for each $f \in \mathcal{B}$, $||T_a f|| \le m\, ||f||_1$.

Proof

(i) Let $\epsilon > 0$. It follows from the fact that the σ-algebra $\mathcal{E} \times \mathcal{E}$ is generated by the set $\mathcal{R}$ of rectangles that the measure $\nu \otimes \nu$ is lower regular with respect to $\mathcal{R}$. From this it can be proved that there exist ν-integrable functions, $\beta_\ell^{(j)}$, $j = 1, 2$, $\ell = 1, \ldots, k$, such that, with $a_\epsilon(x,y) = \sum_{\ell=1}^k \beta_\ell^{(1)}(x) \beta_\ell^{(2)}(y)$, we have

$$\int\int |a(x,y) - a_\epsilon(x,y)| d\nu(x) d\nu(y) < \epsilon.$$

Denote by $\mathcal{F}$ the subspace of the linear space of ν-integrable functions spanned by the functions $\beta_\ell^{(1)}$, $\ell = 1, \ldots, k$. Set, for $f \in \mathcal{B}$ and $x \in E$, $V_\epsilon f(x) = \int a_\epsilon(x,y) f(y) d\nu(y)$. V_ϵ is a linear operator from $\mathcal{B}$ to $\mathcal{F}$, and, for $f \in \mathcal{B}$, we have

$$||V_\epsilon f||_1 \le \left(\sum_{\ell=1}^k ||\beta_\ell^{(1)}||_1 \, ||\beta_\ell^{(2)}||_1 \right) ||f||,$$

and

$$||T_a f - V_\epsilon f||_1 \le \int\int |a(x,y) - a_\epsilon(x,y)| \, |f(y)| d\nu(x) d\nu(y) < \epsilon ||f||.$$

Denote by $(F, ||\cdot||_1)$ the normed linear space obtained by identifying two functions of $\mathcal{F}$ which are equal ν-almost everywhere, and by Π the canonical embedding of $\mathcal{F}$ onto F. The set $\Pi V_\epsilon(U)$ is a bounded subset of a finite dimensional normed linear space, thus it is totally bounded. It follows that $V_\epsilon(U)$ is $||\cdot||_1$-totally bounded. Consequently, there exist $f_\ell \in U$, $\ell = 1, \ldots, r$, such that, for any $f \in U$, we have $||V_\epsilon f - V_\epsilon f_j||_1 < \epsilon$, for a suitable j. Hence

$$||T_a f - T_a f_j||_1 \le ||T_a f - V_\epsilon f||_1 + \epsilon + ||V_\epsilon f_j - T_a f_j||_1 < 3\epsilon.$$

This means that $T_a(U) \subset \cup_{j=1}^r B_1(T_a f_j, 3\epsilon)$. So $T_a(U)$ is $\|\cdot\|_1$-totally bounded.

(ii) follows from $|T_a f(x)| \le \int |a(x,y)| \, |f(y)| d\nu(y) \le m \int |f(y)| d\nu(y)$. []

Let us prove point *(a)* of the proposition.

Suppose that a is bounded by m, then it follows from Lemma III.1 *(ii)* that, for any $f \in \mathcal{B}$ and $r > 0$, we have $T_a\big(B_1(f,r)\big) \subset B(T_a f, mr)$. Since $T_{a'}(U)$ is $\|\cdot\|_1$-totally bounded, we deduce that $T_a\big(T_{a'}(U)\big)$ is $\|\cdot\|$-totally bounded. As $(\mathcal{B}, \|\cdot\|)$ is a Banach space, this means that $T_a T_{a'}$ is compact.

Assume now that a is only in $\mathcal{H}_\nu^{\mathcal{C}}$. For any $k \ge 1$, we set $a_k = a\, 1_{\{|a| \le k\}}$. Because of the uniform integrability, we have

$$\lim_k ||T_a - T_{a_k}|| \leq \lim_k \sup_{x \in E} \int \Big| a(x,y) - a_k(x,y)| \Big| d\nu(y) = 0.$$

This implies that $\lim_k T_{a_k} T_{a'} = T_a T_{a'}$. Since, for any $k \geq 1$, $T_{a_k} T_{a'}$ is a compact operator of the Banach space $(\mathcal{B}, \| \cdot \|)$, we conclude that $T_a T_{a'}$ is compact. []

Proof of (b)

For $f \in \mathcal{B}$ and $x \in E$,

$$(S_\chi)^k T_a f(x) = \int_E S(x, dx_1)\chi(x, x_1) \int_E S(x_1, dx_2)\chi(x_1, x_2) \dots$$
$$\dots \int_E S(x_{k-1}, dx_k)\chi(x_{k-1}, x_k) \int_E a(x_k, y) f(y) d\nu(y).$$

By Fubini's theorem, the above iterated integrals can be written $T_{a_k} f(x)$, with

$$a_k(x,y) = \int_E S(x, dx_1)\chi(x, x_1) \int_E S(x_1, dx_2)\chi(x_1, x_2) \dots$$
$$\dots \int_E S(x_{k-1}, dx_k)\chi(x_{k-1}, x_k) a(x_k, y).$$

As S is positive, we deduce the inequality

$$|a_k(x,y)| \leq \|\chi\|^k \textstyle\int_E S^k(x, dx_k)|a(x_k, y)|.$$

Let B be any element of $\mathcal{E}$, we have

$$\int_B |a_k(x,y)| d\nu(y) \leq \|\chi\|^k \int_E S^k(x, dx_k) \int_B |a(x_k, y)| d\nu(y)$$
$$\leq \|\chi\|^k \sup_{x \in E} S^k(x, E) \sup_{x_k \in E} \int_B |a(x_k, y)| d\nu(y).$$

So a_k like a is in $\mathcal{H}_\nu$. []

End of the proof of Theorem III.1

For convenience, we now set, for $k \geq 1$, $Q_\chi^k = (Q_\chi)^k$ and $S_\chi^k = (S_\chi)^k$.

A. *Case $\ell = 1$.*

Lemma III.2

For $n \geq 1$,

$$Q_\chi^n = K_n + \sum_{k=0}^{n-1} S_\chi^k T_{\alpha\chi} S_\chi^{n-1-k} + S_\chi^n,$$

where K_n is a compact operator of $\mathcal{B}$.

Proof

The assertion is clearly true for $n = 1$. Assume it holds at order n.We have

$$Q_\chi^{n+1} = K_n Q + \sum_{k=0}^{n-1} S_\chi^k T_{\alpha\chi} S_\chi^{n-1-k} T_{\alpha\chi} + S_\chi^n T_{\alpha\chi} + \sum_{k=0}^{n-1} S_\chi^k T_{\alpha\chi} S_\chi^{n-k} + S_\chi^{n+1}.$$

The operator $K_n Q$ is compact. The function $\alpha\chi$ is in $\mathcal{H}_\nu^{\mathbb{C}}$, so we deduce from Proposition III.1, that, for $k = 0, \dots, n-1$, $\big(S_\chi^k T_{\alpha\chi}\big)\big(S_\chi^{n-1-k} T_{\alpha\chi}\big)$ is compact. Hence the assertion at order $n+1$. []

Let $\rho > r(S_\chi)$, there exists $c \in R_+$ such that, for each $\ell \geq 0$, $||S_\chi^\ell|| \leq c\,\rho^\ell$. With the notations of the above lemma, we get

$$\|Q_\chi^n - K_n\| \le \|T_{\alpha\chi}\| c^2\, n\rho^{n-1} + c\,\rho^n.$$

It follows that $r_e(Q_\chi) = \lim_n \big(\inf\{\|Q^n - V\| : V \in \mathcal{K}(\mathcal{B})\}\big)^{1/n} \le \rho$. Finally, getting rid of ρ, we get

$$r_e(Q_\chi) \le r(S_\chi) \le \|\chi\| r(S)$$

as claimed.

B. *Case $\ell \ge 2$.*

As pointed out in [Her], for any positive $f \in \mathcal{B}$ and $x \in E$, we have $|(Q_\chi)^\ell f(x)| \le \|\chi\|^\ell Q^\ell f(x)$, so that the measure $(Q_\chi)^\ell(x,\cdot)$ is absolutely continuous with respect to $Q^\ell(x,\cdot)$. Consequently, if $\mathcal{E}$ is countably generated, there exists $\chi_\ell \in \mathcal{X}$ such that $\|\chi_\ell\| \le \|\chi\|^\ell$ and $(Q_\chi)^\ell = (Q^\ell)_{\chi_\ell}$; see Lemma V.4 in the Appendix.

When applied to Q^ℓ and χ_ℓ, the result of case **B** gives

$$r_e((Q^\ell)_{\chi_\ell}) \le \|\chi_\ell\|\, r(S) \le \|\chi\|^\ell r(S),$$

hence, using *(i)* of Corollary II.1, we get

$$r_e(Q_\chi) = r_e((Q_\chi)^\ell)^{1/\ell} = r_e((Q^\ell)_{\chi_\ell})^{1/\ell} \le \|\chi\| r(S)^{1/\ell}.$$

This completes the proof of Theorem III.1. []

Remarks III.1

The assertion *(a)* in Proposition III.1 is established in [Wu2] Lemma 9.1 as a consequence of some general results on Banach lattices. The proof above is complete and elementary, giving a better understanding of what makes things work.

The **Doeblin's condition** known for Markov kernels may be adapted to provide an upper bound for $r_e(Q)$.

Definition III.2

For $\nu \in \mathcal{P}$ and $Q \in \mathcal{T}(E,\mathcal{E})$, we set

$$\Delta_\nu(Q) = \limsup_{A\in\mathcal{E},\nu(A)\to 0} \big(\sup_{x\in E} Q(x,A)\big).$$

Corollary III.2

Suppose that the σ-algebra $\mathcal{E}$ is countably generated, and that $Q \in \mathcal{T}(E,\mathcal{E})$ is such that there exist an integer ℓ and a probability distribution ν for which $\Delta_\nu(Q^\ell)^{1/\ell} < r(Q)$.

Then

(i) the operator Q is quasi-compact and $r_e(Q) \le \Delta_\nu(Q^\ell)^{1/\ell}$,

(ii) if $\chi \in \mathcal{X}$ is such that $\|\chi\|\,\Delta_\nu(Q^\ell)^{1/\ell} < r(Q_\chi)$, the operator Q_χ is quasi-compact and $r_e(Q_\chi) \le \|\chi\|\,\Delta_\nu(Q^\ell)^{1/\ell}$.

Proof

Let ρ, $\Delta_\nu(Q^\ell)^{1/\ell} < \rho < r(Q)$. Then Q verifies the **Doeblin's condition** :

$(\mathcal{D})$ there exists $\eta > 0$, such that

$$\forall A \in \mathcal{E},\ \ \big(\ \nu(A) \le \eta\ \big)\ \Rightarrow\ \big(\ \forall x \in E,\ Q^\ell(x,A) \le \rho^\ell\ \big).$$

Lemma III.4

Condition $(\mathcal{D})$ implies that there exists $\alpha \in \mathcal{H}_\nu$ such that $S = Q^\ell - T_{\nu,\alpha} \ge 0$ and $\|S\|^{1/\ell} \le \rho$.

Assume this lemma for a while and apply Theorem III.1. We obtain $r_e(Q_\chi) \le \rho\,\|\chi\|$, and hence, getting rid of ρ, $r_e(Q_\chi) \le \|\chi\|\,\Delta_\nu(Q^\ell)^{1/\ell}$ as claimed.

Proof of Lemma III.4

Using differentiation of measures (see Lemma V.4), we get $Q^\ell = T_{\nu,\alpha'} + S'$, where $\alpha' \ge 0$ and $\alpha' \in \mathcal{I}_\nu$, while, for any $x \in E$, there exists $C_x \in \mathcal{E}$ such $\nu(C_x) = 0$ and $S'(x, C_x^c) = 0$. We cannot assert that the functions $\alpha'(x,\cdot)$, $x \in E$, are uniformly ν-integrable.

Set $\alpha = \alpha'\,1_{\left\{\alpha' \le \eta^{-1}\|Q^\ell\|\right\}}$ and, for each $x \in E$, $L_x = \{y : y \in C_x^c, \alpha'(x,y) > \eta^{-1}\|Q^\ell\|\}$. We have $Q^\ell = T_{\nu,\alpha} + S$, with $S(x,A) = S'(x,A) + Q^\ell(x, L_x \cap A) = Q^\ell(x, A \cap (C_x \cup L_x))$. The function α is bounded, so it is in $\mathcal{H}_\nu$. From the inequality

$$\|Q^\ell\| \ge Q^\ell(x, L_x) \ge \int_{L_x} \alpha'(x,y) d\nu(y) \ge \eta^{-1}\|Q^\ell\|\,\nu(L_x),$$

we get $\nu(C_x \cup L_x) = \nu(L_x) \le \eta$. By assumption, this implies $Q^\ell(x, C_x \cup L_x) \le \rho^\ell$, and it follows that $||S|| = \sup_{x\in E} Q^\ell(x, C_x \cup L_x) \le \rho^\ell$. []

III.2 Lower bounds and formulas

To state a converse to the assertion *(i)* of Theorem III.1, we need the following elements.

Definition III.3

We denote by $\mathcal{M}$ the space of bounded complex measures on $(E,\mathcal{E})$. For $\mu \in \mathcal{M}$, we set $\|\mu\| = v(\mu)(E)$, where $v(\mu)$ is the total variation of μ.

A function K from $E \times \mathcal{E}$ to $\mathbb{C}$ is a bounded kernel if

(i) $\forall A \in \mathcal{E}$, $K(\cdot, A)$ *is $\mathcal{E}$-measurable,*

(ii) $\forall x \in E$, $K(x,\cdot)$ *is a bounded complex measure on $(E,\mathcal{E})$,*

(iii) $\sup_{(x,A)\in E\times\mathcal{E}} |K(x,A)| < +\infty$.

We denote by $\mathcal{N}(E,\mathcal{E})$ the space of bounded kernels on $(E,\mathcal{E})$.

About $\mathcal{M}$ recall that, as a corollary to the Vitali-Hahn-Sachs' Theorem, [DS], III-7-4, if $(\mu_n)_{n\ge1}$ is a sequence in $\mathcal{M}$, such that, for any $A \in \mathcal{E}$, the sequence $(\mu_n(A))_{n\ge1}$ converges, then the limit set function μ is in $\mathcal{M}$. In particular, it follows that $\mathcal{M}$ is a Banach space.

Just as was done in Section I for positive kernels, we associate with a bounded kernel a bounded operator on $\mathcal{B}$. We still use the notation $\mathcal{N}(E,\mathcal{E})$ to denote the space of these operators.

Theorem III.2

Suppose that $Q \in \mathcal{T}(E,\mathcal{E})$ is quasi-compact on $\mathcal{B}$, and let the real number ρ be such that $r_e(Q) < \rho < r(Q)$.

Then

(i) there exist bounded kernels L and N such that $Q = L + N$, $LN = NL = 0$, $r(N) < \rho$, L has a finite dimensional range and its non zero eigenvalues have modulus $\ge \rho$,

there exists an integer ℓ_ρ such that, for any $\ell \ge \ell_\rho$, there exists $\nu \in \mathcal{P}$ such that

(ii) $\Delta_\nu(Q^\ell) \le \rho^\ell$,

(iii) assume moreover that $\mathcal{E}$ is countably generated, then there exists a $\alpha \in \mathcal{H}_\nu$ such that $S = Q^\ell - T_{\nu,\alpha} \geq 0$ and $\|S\|^{1/\ell} < \rho$.

Proof

(i) is a consequence of Lemma II.5, since

Lemma III.5

$\mathcal{N}(E,\mathcal{E})$ is a closed subalgebra of $\mathcal{L}(\mathcal{B})$.

Proof of Lemma III.5

The fact that $\mathcal{N}(E,\mathcal{E})$ is an algebra is of constant use. Let $(K_n)_{n\geq 1}$ be a sequence in $\mathcal{N}(E,\mathcal{E})$ and $T \in \mathcal{L}(\mathcal{B})$ such that $\lim_n \|K_n - T\| = 0$. For any $x \in E$ and $A \in \mathcal{E}$, we have $\lim_n K_n(x,A) = T1_A(x)$. It is easily check, using the Vitali-Hahn-Sachs' Theorem, that $T \in \mathcal{T}(E,\mathcal{E})$. []

(ii) is now deduced from *(i)*. We choose ℓ_ρ such that, for any $\ell \geq \ell_\rho$, $\|N^\ell\| < \rho^\ell$. Let $(f_1,\ldots,f_s)$ be a basis of $F = L^\ell(\mathcal{B})$. Since $\{f : f \in F, \forall x \in E, \int f d\delta_x = f(x) = 0\} = \{0\}$, there exists $(x_1,\ldots,x_s) \in E^s$ such that $(\delta_{x_1},\ldots,\delta_{x_s})$ is a basis of the dual space F^* of F. It follows that there exists a $s \times s$ complex matrix $[a_{k,j}]_{k,j=1}^s$, such that setting $\mu_k = \sum_{j=1}^s a_{k,j}\delta_{x_j}$ we have, for $k,m = 1,\ldots,s$, $\int f_m d\mu_k = \delta_{m,k}$, i.e. $(\mu_k)_{k=1}^s$ is the dual basis of $(f_k)_{k=1}^s$. So, for any $f \in \mathcal{B}$, we can write

$$L^\ell f = \sum_{k=1}^s \Big(\int L^\ell f d\mu_k\Big) f_k = \sum_{k=1}^s \sum_{j=1}^s a_{k,j} L^\ell f(x_j) f_k.$$

Choose a probability measure ν with respect to which each bounded measure $L^\ell(x_j,\cdot)$, $j = 1,\ldots,s$ is absolutely continuous, and denote by ζ_j, $j = 1,\ldots,s$, versions of the corresponding Radon-Nikodym derivatives. The above formula becomes, for $x \in E$,

$$L^\ell f(x) = \sum_{k=1}^s \sum_{j=1}^s a_{k,j}\Big(\int f(y)\zeta_j(y)d\nu(y)\Big) f_k(x) = \int \beta(x,y) f(y) d\nu(y),$$

with $\beta(x,y) = \displaystyle\sum_{k=1}^s \sum_{j=1}^s a_{k,j} f_k(x)\zeta_j(y)$. Since the functions f_k are bounded, the functions $\beta(x,\cdot)$, $x \in E$, are uniformly ν-integrable. It follows that

$$\limsup_{A\in\mathcal{E},\nu(A)\to 0} \sup_{x\in E} \int_A \beta(x,y)d\nu(y) = 0.$$

So

$$\Delta_\nu(Q^\ell) = \limsup_{A\in\mathcal{E},\nu(A)\to 0} \big(\sup_{x\in E} Q^\ell(x,A)\big) \leq \limsup_{A\in\mathcal{E},\nu(A)\to 0} \sup_{x\in E} |N^\ell(x,A)| \leq \|N^\ell\| < \rho,$$

that is *(ii)*.

Finally, Lemma III.4 shows that *(ii)* implies *(iii)*. So Theorem III.2 is proved. []

Collecting the results of this section, we obtain several formulas for the essential spectral radius.

Theorem III.3

Suppose that $\mathcal{E}$ is countably generated and let $Q \in \mathcal{T}(E,\mathcal{E})$.

(i) We have $r_e(Q) = \inf\limits_{\ell\geq 1,\nu\in\mathcal{P}} \Delta_\nu(Q^\ell)^{1/\ell}$.

(ii) Set $\mathcal{K}^* = \{T_{\nu,\alpha} : \nu \in \mathcal{P}, \alpha \in \mathcal{H}_\nu\}$. *Then*

$$\begin{aligned} r_e(Q) &= \inf\big\{\|Q^\ell - T\|^{1/\ell} : \ell \geq 1,\ T \in \mathcal{K}^*,\ Q^\ell - T \geq 0\big\} \\ &= \inf\big\{r(Q^\ell - T)^{1/\ell} : \ell \geq 1,\ T \in \mathcal{K}^*,\ Q^\ell - T \geq 0\big\}. \end{aligned}$$

Proof

(i) Set $\overline{\Delta}(Q) = \inf_{\ell\geq 1,\nu\in\mathcal{P}} \Delta_\nu(Q^\ell)^{1/\ell}$. From Corollary III.2, we know that $r_e(Q) \leq \overline{\Delta}(Q)$. Conversely, Theorem III.2-*(iii)* asserts that $\rho > r_e(Q)$ implies $\rho > \overline{\Delta}(Q)$, so that $r_e(Q) \geq \overline{\Delta}(Q)$.

(ii) Set $\Theta(Q) = \inf\big\{\|Q^\ell - T\|^{1/\ell} : \ell \geq 1, T \in \mathcal{K}^*, Q^\ell - T \geq 0\big\}$
$\Theta_r(Q) = \inf\big\{r(Q^\ell - T)^{1/\ell} : \ell \geq 1, T \in \mathcal{K}^*, Q^\ell - T \geq 0\big\}$.
We have $\Theta_r(Q) \leq \Theta(Q)$. If $\rho > \Theta_r(Q)$, there exist $\ell \geq 1$ and $T \in \mathcal{K}^*$ such that $S = Q^\ell - T \geq 0$ and $r(S)^{1/\ell} < \rho$. From Theorem III.1, we get $r_e(Q) \leq r(S)^{1/\ell} < \rho$. So $r_e(Q) \leq \Theta_r(Q)$. Conversely, for $\rho > r_e(Q)$, Theorem III.2-*(iii)* asserts that there exist ℓ and $T \in \mathcal{K}^*$ such that $Q^\ell - T \geq 0$ and $\|Q^\ell - T\|^{1/\ell} < \rho$. So $\Theta(Q) \leq r_e(Q)$. []

The kernel Q has a canonical action on the Banach space $\mathcal{M}$ of bounded complex measures. The preceding results provide tools to compare the essential radius of the two actions of Q. Duality may then be used to study the action of Q on some subspaces of $\mathcal{B}$.

Corollary III.3

Let $Q \in \mathcal{T}(E,\mathcal{E})$. Setting, for each $\mu \in \mathcal{M}$ and $A \in \mathcal{E}$,
$$\big(Q^*\mu\big)(A) = \textstyle\int Q(x,A)d\mu(x),$$
we define a bounded linear operator Q^* on $(\mathcal{M}, \|\cdot\|)$.

We have $\|Q^*\| = \|Q\|$, $r(Q^*) = r(Q)$, and $r_e(Q^*) = r_e(Q)$.

Proof

More generally, a * operator can be associated to a kernel $K \in \mathcal{N}(E,\mathcal{E})$. The equality $\|K^*\| = \|K\|$ follows from the fact that, for $f \in \mathcal{B}$ and $\mu \in \mathcal{M}$, we have
$\|f\| = \sup\{\big|\int f d\mu_1\big| : \mu_1 \in \mathcal{M}, \|\mu_1\| \leq 1\}$, $\|\mu\| = \sup\{\big|\int f_1 d\mu\big| : f_1 \in \mathcal{B}, \|f_1\| \leq 1\}$.
Consequently $r(K^*) = r(K)$.

Assume that Q is quasi-compact on $\mathcal{B}$, and let $r_e(Q) < \rho < r(Q)$. Applying the assertion *(i)* of Theorem III.2, we get $Q^* = L^* + N^*$, $L^*N^* = N^*L^* = 0$, $r(N^*) = r(N) < \rho$, and it is easily verified that L^* has a finite rank. So $\rho \geq r_e(Q^*)$. We have $r_e(Q^*) \leq r_e(Q)$.

Conversely, assume that Q^* is quasi-compact on $\mathcal{M}$, and let $r_e(Q^*) < \rho < r(Q^*)$. The set $\mathcal{N}(E,\mathcal{E})^* = \{Q^* : Q \in \mathcal{N}(E,\mathcal{E})\}$ is a subalgebra of the Banach algebra $\mathcal{L}(M)$. Since the mapping $Q \to Q^*$ from $\mathcal{L}(\mathcal{B})$ to $\mathcal{L}(M)$ preserves the norm and $\mathcal{N}(E,\mathcal{E})$ is closed, $\mathcal{N}(E,\mathcal{E})^*$ is closed. Applying Lemma II.2, we can assert the existence of Λ and $\Upsilon \in \mathcal{N}(E,\mathcal{E})^*$, such that $Q^* = \Lambda + \Upsilon$, $\Lambda\Upsilon = \Upsilon\Lambda = 0$, $r(\Upsilon) < \rho$, Λ has a finite rank. But there exist L and $N \in \mathcal{N}(E,\mathcal{E})$ such that $L^* = \Lambda$ and $N^* = \Upsilon$. One can check that the properties of L and N ensure that $r_e(Q) \leq \rho$. So $r_e(Q) \leq r_e(Q^*)$. []

Corollary III.4

Let $\mathcal{F}$ be a closed subspace of $\mathcal{B}$ such that,

$$\text{for any } \mu \in \mathcal{M}, \quad v(\mu)(E) = \sup\{|\int f d\mu| : f \in \mathcal{F}, \|f\| = 1\}.$$

Then, if $Q \in \mathcal{T}(E, \mathcal{E})$ and $\mathcal{F}$ is Q-invariant, we have $r_e(Q_{|\mathcal{F}}) = r_e(Q)$.

Suppose E is a metric space and $\mathcal{E}$ is its Borel σ-field. Then the subspace $\mathcal{C}$ of bounded continuous functions on E is closed in $\mathcal{B}$ and verifies the condition stated above for the computation of the norms of measures. Consequently, if Q is a **Feller kernel**, i.e $Q(\mathcal{C}) \subset \mathcal{C}$, we have $r_e(Q_{|\mathcal{C}}) = r_e(Q)$.

Proof

Set $\widetilde{Q} = Q_{|\mathcal{F}}$. From Corollary II.1-*(ii)*, we have $r_e(\widetilde{Q}) \leq r_e(Q)$. Let $\mathcal{F}'$ be the topological dual space of $\mathcal{F}$ and $\widetilde{Q}'$ be the adjoint of $\widetilde{Q}$. By point *(iv)* of Corollary II.1, $r_e(\widetilde{Q}') = r_e(\widetilde{Q})$. Any $\mu \in \mathcal{M}$ defines an element of $\mathcal{F}'$, whose norm is $v(\mu) = |\mu|(E)$ by assumption. As $(\mathcal{M}, \|\cdot\|)$ is a Banach space, it is a closed subspace of $\mathcal{F}'$. Hence $r_e(Q) = r_e(Q^*) \leq r_e(\widetilde{Q}') = r_e(\widetilde{Q})$. []

III.3 Link with the weak compactness in $\mathcal{M}$

Recall that a subset M of $\mathcal{M}$ is said to be weakly sequentially compact if, for any sequence $(\mu_n)_{n\geq 1}$ in M, there exist $\mu \in \mathcal{M}$ and $(n_k)_{k\geq 1}$ such that, for any $\phi \in \mathcal{M}'$, $\lim_k \langle \phi, \mu_{n_k} \rangle = \langle \phi, \mu \rangle$. According to the Eberlein-Šmulian' Theorem [DS] V.6.1, this is equivalent to the fact that M is conditionally compact in the $\sigma(\mathcal{M}, \mathcal{M}')$-topology. This compactness property has several characterizations that we now recall, see [DS] Theorems IV.9-1 and 2.

Theorem III.4

For $M \subset \mathcal{M}$, the three following assertions are equivalent :

(i) M is weakly sequentially compact,

(ii) M is bounded and there exist a probability measure ν on E such that absolutely continuity with respect to ν is uniform on the set M,

(iii) M is bounded and σ-additivity is uniform on the set M.

The σ-additivity is said to be uniform on M, if, for any sequence $(A_n)_{n\geq 1}$ in $\mathcal{E}$ which decreases to $\emptyset$, we have $\lim_n \sup_{\mu\in M} \mu(A_n) = 0$. Otherwise, it is easily seen, that, if any measure in M is absolutely continuous with respect to a probability measure ν, uniform absolutely continuity is equivalent to the uniform ν-integrability of the set of Radon-Nikodym derivatives $\{\frac{d\mu}{d\nu} : \mu \in M\}$.

The above theorem yields a characterization of the class $\mathcal{K}^*$ of kernels defined in Theorem III.3.

Lemma III.6

Assume that $\mathcal{E}$ is countably generated. Let $U_{\mathcal{M}}$ be the closed unit ball of $\mathcal{M}$.
For $Q \in \mathcal{T}(E, \mathcal{E})$, we have $Q \in \mathcal{K}^*$ if and only if $Q^*(U_{\mathcal{M}})$ is weakly sequentially compact in $\mathcal{M}$, i.e. Q^* is a weakly compact operator of $\mathcal{M}$.

Proof

It is based on the equivalence of points *(i)* and *(ii)* in Theorem III.4.

If $Q \in \mathcal{K}^*$, there exist $\nu \in \mathcal{P}$ and $\alpha \in \mathcal{H}_\nu$ such that $Q = T_{\nu,\alpha}$. Consequently the set $\{Q(x,\cdot) : x \in E\}$ is uniformly absolutely continuous with respect to ν. It follows that this property also holds for $Q^*(U_{\mathcal{M}}) = \{\int d\mu(x)Q(x,\cdot) : \mu \in \mathcal{M}, \|\mu\| \leq 1\}$. Hence $Q^*(U_{\mathcal{M}})$ is weakly sequentially compact in $\mathcal{M}$.

Conversely, suppose that $\{Q(x,\cdot) : x \in E\}$ is weakly sequentially compact in $\mathcal{M}$. There exist $\nu \in \mathcal{P}$ such that the absolute continuity with respect to ν is uniform over $\{Q(x,\cdot) : x \in E\}$. We have the Radon-Nikodym decomposition $Q(x,A) = \int_A \alpha(x,y)d\nu(x)$, $(x,A) \in E \times \mathcal{E}$, where α is a positive measurable function on $E \times E$. The uniform absolute continuity claimed above is just the uniform ν-integrability of functions $\alpha(x,\cdot)$, $x \in E$. So $Q \in \mathcal{K}^*$. []

Thus formulas *(ii)* of Theorem III.3 mean that the essential spectral radius of a bounded positive kernel Q is related to the lower approximation of Q by positive kernels whose action on the space of bounded measures is weakly sequentially compact. This may be compared to the formula of Theorem II.1 which shows that, in the abstract context, the essential spectral radius of an operator Q is linked to the approximation of Q by compact operators.

Let $Q \in \mathcal{T}(E,\mathcal{E})$. For $\nu \in \mathcal{P}$, $\Delta_\nu(Q) = \limsup_{A\in\mathcal{E},\nu(A)\to 0}\big(\sup_{x\in E} Q(x,A)\big)$ is a measure of the non uniform absolute continuity with respect to ν over $\{Q(x,\cdot) : x \in E\}$. By Theorem III.4, $\inf_{\nu\in\mathcal{P}} \Delta_\nu(Q)$ is a measure of the non weak sequential compactness of $\{Q(x,\cdot) : x \in E\}$. So the formulas *(i)* and *(ii)* of Theorem III.3 have a similar heuristic. One is based on an operator formulation, while the other uses a set theoretical frame. Of course these points of view are intimately related as shown by the proofs of this section. These remarks lead to introduce several measures of the non weak sequential compactness for a subset M of the cone $\mathcal{M}_+$ of positive measures on $(E,\mathcal{E})$.

Definition III.4

Let $\mathcal{C}_{ws}$ be the collection of all weakly sequentially compact subsets of $\mathcal{M}_+$. For $M \subset \mathcal{M}_+$, we set

$$\Gamma(M) = \inf\{\sup_{\mu\in M} d(\mu,K) : K \in \mathcal{C}_{ws}\}, \quad \textit{with} \;\; d(\mu,K) = \inf_{\nu\in K} \|\mu - \nu\|,$$

$$\Delta(M) = \inf_{\nu\in\mathcal{P}} \Delta_\nu(M), \quad \textit{with} \;\; \Delta_\nu(M) = \limsup_{A\in\mathcal{E},\nu(A)\to 0} \big(\sup_{\mu\in M} \mu(A)\big),$$

$$\Lambda(M) = \sup\big\{\lim_n \sup_{\mu\in M} \mu(A_n) : n \geq 1,\; A_n \in \mathcal{E},\; (A_n)_n \downarrow \emptyset\big\}.$$

The number $\Gamma(M)$ measures the distance of the set M to the class $\mathcal{C}_{ws}$, while $\Delta(M)$ and $\Lambda(M)$ measure, respectively, the non uniform absolute continuity and the non uniform σ-additivity over M. Notice that Δ_ν has already been defined as a function on $\mathcal{T}(E,\mathcal{E})$ (Definition III.2), but this will not be confusing, in fact $\Delta_\nu(Q) = \Delta_\nu\{Q(x,\cdot) : x \in E\}$.

Proposition III.1

Let M be a bounded subset of $\mathcal{M}_+$.

(i) *We have* $\Gamma(M) = \Delta(M) \geq \Lambda(M)$.

For $\nu \in \mathcal{P}$, set $\partial_\nu(M) = \lim_{k\to+\infty} \big(\sup_{\mu\in M} \mu\{x : x \in E, \dfrac{d\mu}{d\nu}(x) \geq k\}\big)$. *Then*

(ii) if any measure in M is absolutely continuous with respect to ν, we have
$$\Delta(M) = \Delta_\nu(M) = \partial_\nu(M) = \Lambda(M),$$
(iii) more generally, we have $\partial_\nu(M) \le \Delta_\nu(M) \le \partial_\nu(M) + \sup_{\mu\in M} \mu_{\perp\nu}(E)$, where $\mu_{\perp\nu}$ is the singular part of μ in the Lebesgue' decomposition of μ with respect to ν.

Proof

Let $t > \Gamma(M)$. There exist $K \in \mathcal{C}_{ws}$ such that, for any $\mu \in M$, we have $d(\mu, K) < t$. From Theorem III.4, there exists a $\nu \in \mathcal{P}$ such that absolute continuity with respect to ν is uniform on K. For any $\mu \in M$, there exists a μ_1 in K such that $\|\mu - \mu_1\| < t$. It follows that
$$\lim_{\nu(A)\to 0} \sup_{\mu\in M} \mu(A) \le t + \lim_{\nu(A)\to 0} \sup_{\mu_1\in K} \mu_1(A) = t.$$
We get $t \ge \Delta_\nu(M) \ge \Delta(M)$. So $\Gamma(M) \ge \Delta(M)$.

Let $t > \Delta(M)$. If $\nu \in \mathcal{P}$ is such that $\Delta_\nu(M) < t$, there exists a η such that $\nu(A) \le \eta$ implies that, for any $\mu \in M$, $\mu(A) < t$. Then, since M is bounded the proof of Lemma III.4 can be adapted to show that, for all $\mu \in M$, $\mu = \alpha_\mu.\nu + \sigma_\mu$, where the functions α_μ, $\mu \in M$, are uniformly bounded and $\|\sigma_\mu\| < t$. From Theorem III.4, $\{\alpha_\mu.\nu : \mu \in M\} \in \mathcal{C}_{ws}$, so $\Gamma(M) < t$ and hence $\Delta(M) \ge \Gamma(M)$. We have thus prove that $\Gamma(M) = \Delta(M)$.

As, for any sequence $(A_n)_{n\ge 1}$ in $\mathcal{E}$ decreasing to $\emptyset$ and $\nu \in \mathcal{P}$, we have $\lim_n \nu(A_n) = 0$, we see that, for any $\nu \in \mathcal{P}$, $\Delta_\nu(M) \ge \Lambda(M)$. So $\Delta(M) \ge \Lambda(M)$.

This proves *(i)*.

We now assume that any measure in M is absolutely continuous with respect to ν.

The equality $\Delta_\nu(M) = \Lambda(M)$ follows from *(i)* when $\Delta_\nu(M) = 0$. Let $0 < t < \Delta_\nu(M)$. Then, for each $n \ge 1$, there exist $A_n \in \mathcal{E}$ and $\mu_n \in M$ such that $\nu(A_n) \le 2^{-n}$ and $\mu_n(A_n) > t$. Set $B_n = \cup_{k\ge n} A_k$, $A = \cap_{n\ge 1} B_n = \limsup_n A_n$, and $C_n = B_n \backslash A$. The sequence $(C_n)_{n\ge 1}$ decreases to $\emptyset$. Since $\nu(A) = 0$, we have $\mu_n(C_n) = \mu_n(B_n) \ge \mu_n(A_n) > t$. It follows that $\Lambda(M) > t$. So $\Lambda(M) \ge \Delta_\nu(M)$. Finally, using *(i)*, we get $\Lambda(M) = \Delta_\nu(M) = \Delta(M)$.

The relation $\Delta_\nu(M) = \partial_\nu(M)$ is obtained by a straightforward adaptation of the standart arguments used in the proof of the equivalence of the uniform ν-integrability of a set of functions $\{f_i : i \in I\}$ and of the uniform absolute continuity of the set of measures $\{(f_i \cdot \nu) : i \in I\}$ with respect to ν.

Assertion *(iii)* follows easily from the previous ones. []

Corollary III.6

Assume $\mathcal{E}$ is countably generated. For $Q \in \mathcal{T}(E, \mathcal{E})$, we have
$$\begin{aligned} r_e(Q) &= \inf_{\ell\ge 1} \Delta(\{Q^\ell(x,\cdot) : x \in E\})^{1/\ell} \\ &= \inf_{\ell\ge 1} \Gamma(\{Q^\ell(x,\cdot) : x \in E\})^{1/\ell} \ge \bar{\Lambda}(Q) = \inf_{\ell\ge 1} \Lambda(\{Q^\ell(x,\cdot) : x \in E\})^{1/\ell}. \end{aligned}$$

If there exists $\nu \in \mathcal{P}$, $\ell_0 \ge 1$ and positive measurable functions q_ℓ on $E \times E$ such that, for each $\ell \ge \ell_0$ and $x \in E$, we have $Q^\ell(x,\cdot) = q_\ell(x,\cdot)\cdot\nu$, then
$$\begin{aligned} r_e(Q) &= \inf_{\ell\ge 1} \Delta_\nu(\{Q^\ell(x,\cdot) : x \in E\})^{1/\ell} = \bar{\Lambda}(Q) \\ &= \inf_{\ell\ge 1} \lim_{k\to+\infty} \sup_{x\in E} \Big(\int_{\{y:\ q_\ell(x,y)\ge k\}} q_\ell(x,y)d\nu(y)\Big)^{1/\ell}. \end{aligned}$$

Notice that the absolute continuity of Q^ℓ involved in the last assertion holds for any $\ell \geq \ell_0$ as soon as it holds for ℓ_0.

Proof

The first equality is merely a reformulation of Theorem III.3-*(i)*. The other relations are deduced from Proposition III.1. []

Remark III.2

L. Wu [Wu2] has obtained several inequalities for $r_e(Q)$ when E is a polish space. One of these is based on the set function Λ, he denoted β_τ, the number $\Lambda(M) = \beta_\tau(M)$ being considered as a measure of the non compactness of M for the weak topology $\sigma(\mathcal{B}', \mathcal{B})$, instead of $\sigma(\mathcal{M}, \mathcal{M}')$ as here. Set $\Lambda(Q) = \Lambda(\{Q(x,\cdot) : x \in E\})$. Using the Nussbaum's formula associated with the set function γ measuring non compactness , Wu gives a direct proof of the inequality $r_e(Q) \geq \bar{\Lambda}(Q) = \inf_{\ell \geq 1} \Lambda(\{Q^\ell(x,\cdot) : x \in E\})^{1/\ell}$. But nearly all his other results are obtained under an hypothesis he call $(A1)$. This hypothesis implies that there exists an n_0 such that, for any compact set K in E, we have $\Lambda(1_K Q^{n_0}) = 0$. Actually $(A1)$ is very restrictive, since, in the context of our study, assuming that E is a topological space, we only have $\bar{\Lambda}(T) = 0$, for any $T \in \mathcal{K}^*$. Under $(A1)$, Wu has obtained the relation *(i)* of Theorem III.3 with $\leq$ instead of $=$, Corollary 3.6. The crucial point for this is Lemma 9.1, already mentioned in Remark III.1. By the way notice that $\alpha \in \mathcal{H}_\nu$ implies that $T_{\nu,\alpha}$ defines a uniformly integrable operator in $L^\infty(\nu)$, [Wu1]. Otherwise the statements of Section III.3 may be discussed in the more general setting of a positive operator on a Banach lattice, since Theorem III.4 has an analogue in this context, see [M-N] Section 2.5.

An example of non quasi-compactness on $\mathcal{B}$

Let $E = [0,1]$ endowed with its Borel σ-field $\mathcal{E}$. Denote by u a positive measurable function on $[0,1]$ such that, for any $x \in [0,1]$, $u(\frac{x}{2}) + u(\frac{x+1}{2}) = 1$. We associate with u the Markov kernel P defined, for $f \in \mathcal{B}$ and $x \in [0,1]$, by

$$Pf(x) = u(\frac{x}{2})\, f(\frac{x}{2}) + u(\frac{x+1}{2})\, f(\frac{x+1}{2}).$$

This type of Markov kernel has been introduced and studied by J-P. Conze and A. Raugi [CR].

Assume that u is Lipschitz. Then it follows straightforwardly from the Ionescu Tulcea Marinescu's Theorem [ITM] that P has a quasi-compact action on the space $\mathcal{L}ip$ of Lipschitz functions on $[0,1]$, endowed with its canonical norm [CR] ; moreover, its essential spectral radius on this space is $\leq 1/2$, [Hen1], [HenHer].

In the case $u = 1/2$, it is easily checked that, for any $\lambda \in \mathbb{C}$, $|\lambda| < 1$, the function f_λ, defined on $[0,1]$ by $f_\lambda(x) = \sum_{n \geq 1} \lambda^{n-1} \cos(2^n \pi x)$, is a continuous eigenfunction associated to the eigenvalue λ. So P is not quasi-compact when it acts on $\mathcal{B}$ or on the subspace $\mathcal{C}$ of all continuous functions in $\mathcal{B}$. Let us use $\bar{\Lambda}$ to prove that, on $\mathcal{B}$, this assertion holds for any u.

For $x \in [0,1]$, set $C_x = \{\frac{x+k}{2^n} : n \geq 1, k = 0, \ldots, 2^n - 1\}$. For any $\ell \geq 1$, we have $P^\ell(x, C_x) = 1$. If x, $x' \in [0,1]$ are such that 1, x, x' are linearly independent over the field

$\mathbb{Q}$, then $C_x \cap C_{x'} = \emptyset$. Let $(x_n)_{n\geq 0}$ be a sequence of $\mathbb{Q}$-linearly independent elements of $[0,1]$ such that $x_0 = 1$. Setting $A_n = \cup_{k\geq n} C_{x_k}$, $n \geq 1$, we define a decreasing sequence of measurable subsets in $[0,1]$ with $\cap_{n\geq 1} A_n = \emptyset$. Since P is Markov and, for any $\ell \geq 1$ and any $n \geq 1$, we have $P^\ell(x_n, A_n) = 1$, we deduce that $\overline{\Lambda}(P) = 1$. It follows from Corollary III.6 or from Theorem 3.5 (b) in [Wu2] that $r_e(P) = 1$.

IV. QUASI-COMPACTNESS PROPERTIES OF MARKOV KERNELS

IV.1 The case of Markov kernels

All the results of the preceding section directly apply to the action of a Markov kernel on the space $\mathcal{B}$ of bounded functions. We select one consequence of Corollary III.2, which shows how our work generalizes Doeblin's result.

Corollary IV.1

Suppose that the σ-algebra $\mathcal{E}$ is countably generated, and that the Markov kernel P verifies the Doeblin's condition : there exist an integer ℓ, a probability distribution ν, and real numbers η, $0 < \eta$, ρ, $0 \leq \rho < 1$, such that,

$$\forall A \in \mathcal{E},\ \ \big(\ \nu(A) \leq \eta\ \big)\ \ \Rightarrow\ \ \big(\ \forall x \in E,\ P^\ell(x,A) \leq \rho^\ell\ \big).$$

Then the operator P is quasi-compact on $\mathcal{B}$ and $r_e(P) \leq \rho^{1/\ell}$.

We see that Doeblin's result is improved, since, on one hand, quasi-compactness is specified by upper bounds for the essential spectral radius, and, on the other hand, no irreducibility and no aperiodicity condition are required.

From a technical point of view, notice that the main problem to apply Theorem III.1 is to get a suitable upper bound for $r(S)$. In the case of Corollary III.2 which implies Corollary IV.1, this is obtained by using the inequality $r(S) \leq \|S\|$. We shall consider below (Theorem IV.2) cases where this crude estimate is not sufficient, the behaviour of the iterated kernels S^n, $n \geq 0$, has to be taken into account.

Untill now quasi-compactness was only considered for the action of kernels on the space $\mathcal{B}$ of bounded measurable functions. For a Markov kernel P this implies that there exist a finite rank kernel L, and real numbers ρ, $0 \leq \rho < 1$, C, $C \geq 0$, such that we have,

$$\forall A \in \mathcal{E}, \forall x \in E, \quad |P^n(x,A) - L^n(x,A)| \leq C\rho^n.$$

Let now w be a measurable function from E to $[1, +\infty[$, the property ,

$$\forall A \in \mathcal{E}, \forall x \in E, \quad |P^n(x,A) - L^n(x,A)| \leq C\rho^n w(x).$$

is weaker than the preceding one. Indeed approximation of the probability measures $P^n(x,\cdot)$ by the measures $L^n(x,\cdot)$ is not uniform over E but over the level sets of w. To take such facts into account we consider quasi-compactness on the space $\mathcal{B}_w$ of functions bounded with respect to the test function w. We now introduce the frame needed for this study and give a version of Theorem III.1 adapted to this setting.

Let w be a measurable function from $(E,\mathcal{E})$ to $[1,+\infty[$. We denote by $\mathcal{B}_w$ the space of complex valued measurable functions f on $(E,\mathcal{E})$, verifying $\sup_E w^{-1}|f| < +\infty$. Endowed with the norm $\|f\|_w = \sup_E w^{-1}|f|$, $\mathcal{B}_w$ is a Banach space.

Let $Q \in \mathcal{T}(E,\mathcal{E})$. Clearly Q defines a bounded linear operator on $\mathcal{B}_w$ if and only if the function $w^{-1}(Qw)$ is bounded ; in this case, we have $\|Q\|_w = \sup_E w^{-1}(Qw)$. We denote by $r^w(T)$ and $r_e^w(T)$ the spectral radius and the essential spectral radius of a bounded

operator T on $\mathcal{B}_w$. Notice that if w is bounded $\mathcal{B} = \mathcal{B}_w$, the norms $\|\cdot\|_w$ and $\|\cdot\|$ are equivalent, and we are in the frame of the preceding section.

In fact all the results of the preceding section can easily be translated to the present setting. Define the linear application W from $\mathcal{B}$ to $\mathcal{B}_w$ by $Wf = wf$, clearly it is an isometric isomorphism of these Banach spaces. So $Q \in \mathcal{T}(E, \mathcal{E})$ acting on $\mathcal{B}_w$ has the same spectral properties as the conjugate operator $Q^{(w)} = W^{-1}QW$ acting on $\mathcal{B}$. The essential spectral radius of Q is now related to the subclass $\mathcal{K}^{*(w)}$ of $\mathcal{T}(E, \mathcal{E})$ that we now define.

Definition IV.1

$$\mathcal{K}^{*(w)} = \bigcup_{\nu \in \mathcal{P}} \{T_{\nu,\alpha} : \alpha \in \mathcal{H}_\nu^{(w)}\},$$

where $\mathcal{H}_\nu^{(w)}$ is the set of positive measurable functions on $E \times E$ such that the functions $\alpha^{(w)}(x, \cdot) = w^{-1}(x)\alpha(x, \cdot)w(\cdot)$, $x \in E$, are uniformly ν-integrable.

We state what follows from Theorem III.1.

Theorem IV.1

Let $Q \in \mathcal{T}(E, \mathcal{E})$ such that $w^{-1}(Qw)$ is bounded. Assume that there exist $\ell \geq 1$, $\nu \in \mathcal{P}$, and $\alpha \in \mathcal{H}_\nu^{(w)}$ such that $S = Q^\ell - T_{\nu,\alpha} \geq 0$ and $r^w(S)^{1/\ell} < r^w(Q)$.

Then

(i) the operator Q is quasi-compact on $\mathcal{B}_w$ and $r_e^w(Q) \leq r^w(S)^{1/\ell}$,

(ii) assume moreover that either $\ell = 1$, or $\mathcal{E}$ is countably generated, then, if $\chi \in \mathcal{X}$ is such that $\|\chi\| \, r^w(S)^{1/\ell} < r^w(Q_\chi)$, the operator Q_χ is quasi-compact on $\mathcal{B}_w$ and we have $r_e^w(Q_\chi) \leq \|\chi\| \, r^w(S)^{1/\ell}$.

Proof

The conjugate operators

$$Q^{(w)} = W^{-1}QW, \quad T_{\nu,\alpha}^{(w)} = W^{-1}T_{\nu,\alpha}W = T_{\nu,\alpha^{(w)}},$$
$$S^{(w)} = W^{-1}(Q^\ell - T_{\nu,\alpha})W = (Q^{(w)})^\ell - T_{\nu,\alpha^{(w)}},$$

act on $\mathcal{B}$ and verify the hypotheses of Theorem III.1. The claimed properties then follow from the relations

$$r^w(Q_\chi) = r(Q_\chi^{(w)}), \quad r_e^w(Q_\chi) = r_e(Q_\chi^{(w)}),$$
$$r^w(S_{\chi_\ell}) = r(S_{\chi_\ell}^{(w)}) \leq \|\chi_\ell\| \, r(S^{(w)}) = \|\chi\|^\ell \, r^w(S),$$

where χ_ℓ is the function defined in the part B of the proof of Theorem III.1. []

IV.2 A sufficient condition for quasi-compactness on $\mathcal{B}_w$

Let P be a Markov kernel P. We denote by $(X_n)_{n\geq 0}$ an associated Markov chain. For $C \in \mathcal{E}$, we set

$$\sigma_C = \inf\{n : n \geq 0, X_n \in C\}.$$

Theorem IV.2

Assume that the Markov kernel P is such that there exists a non empty subset $C \in \mathcal{E}$ verifying the following conditions :

(a) there exist a measurable function w from E to $[1, +\infty[$, and constants $r_1 > 1$, $\eta \geq 0$ such that

$$Pw \leq r_1^{-1}(w \, 1_{C^c} + \eta \, 1_C).$$

(b) there exist b, $0 < b \leq 1$, $\nu \in \mathcal{P}$ and $\alpha \in \mathcal{H}_\nu^{(w)}$ such that,

$$T_{\nu,\alpha} \text{ is Markov and } \forall x \in E,\ \forall A \in \mathcal{E},\quad P(x,A) \geq b\,1_C(x)\,T_{\nu,\alpha}(x,A).$$

Then P is quasi-compact on $\mathcal{B}_w$ with spectral radius $r^w(P) = 1$.

Moreover, define the Markov kernel P_0 by

$$P_0(x,A) = \frac{1}{1 - b\,1_C(x)}\Big(P(x,A) - b\,1_C(x)\,T_{\nu,\alpha}(x,A)\Big), \quad \text{if } b < 1 \text{ or } x \notin C,$$

$$P_0(x,A) = 1_A(x), \quad \text{if } b = 1 \text{ and } x \in C,$$

and set, for r, $0 \leq r \leq r_1$,

$$h(r) = \sup_{x \in C} \int P_0(x,dy)\, r E_y[r^{\sigma_C}].$$

Then

(i) $h(r_1) < +\infty$ and $r_b = \sup\{r : 0 \leq r \leq r_1,\ h(r) < \frac{1}{1-b}\} > 1$, $\quad(\frac{1}{1-b} = +\infty$ if $b = 1)$.

(ii) $r_e^w(P) \leq r_b^{-1}$,

(iii) if $\chi \in \mathcal{X}$ is such that $\|\chi\|\, r_b^{-1} < r^w(P_\chi)$, the kernel P_χ is quasi-compact on $\mathcal{B}_w$ and $r_e^w(P_\chi) \leq \|\chi\|\, r_b^{-1}$.

If $b = 1$ then $h(r) = r$, so that $r_b = r_1$. The hypotheses of this theorem may be verified for P^ℓ rather than for P, however the properties of the spectral radius and of the essential spectral radius allow to get results for P, see, for example, the proof of Theorem III.1. Finally notice that only the restriction of α to $C \times E$ is meaningfull.

Let us establish the link between the above theorem and already known results. Recall that a set $C \in \mathcal{E}$ is said to be small (with respect to P), if there exist $m \geq 1$, $b > 0$, and a $\nu \in \mathcal{P}$, such that we have $P^m \geq b 1_C \nu$; it is said to be "petite", if it is small with respect to the Markov kernel $\sum_{n \geq 0} 2^{-n} P^n$. Theorem IV.2 implies the following statement due to Nummelin and Tweedie [NuTw], see [MeTw] Chap XV.

Corollary IV.2

Assume that the Markov kernel P is irreducible and aperiodic and that there exists a non empty subset $C \in \mathcal{E}$ verifying the following conditions :

(a') there exist a measurable function w from E to $[1, +\infty]$, and constants $0 < \rho < 1$, $\zeta \geq 0$ such that

$$Pw \leq \rho w + \zeta\, 1_C.$$

(b') C is a "petite" set.

Then the set $E_1 = \{x : x \in E, w(x) < +\infty\}$ is absorbing (i.e. $P 1_{E_1} \geq 1_{E_1}$), P is quasi-compact on $\mathcal{B}_w^{E_1}$, its spectral radius is 1, 1 is the only eigenvalue of modulus 1 and it is simple. Consequently, there exist constants D and $0 \leq \kappa < 1$, such that, for any $n \geq 0$, $f \in \mathcal{B}_w$ and $x \in E$,

$$|P^n f(x) - \pi(f)| \leq D\, \|f\|_w \kappa^n w(x),$$

where π is the unique P-invariant probability distribution.

By $\mathcal{B}_w^{E_1}$ we mean $\mathcal{B}_w$ with E_1 instead of E.

In standard terminology, the function w in *(a')* is said to be a Foster-Lyapounov function associated with P and C. Since when P is irreducible and aperiodic every "petite" set is small, [MeTw] Theorem 5.5.7, hypothesis *(b')* really means that some power of P^m

verifies hypothesis *(b)* of Theorem IV.2 with $\alpha = 1$; such an α clearly belongs to $\mathcal{H}_\nu^{(w)}$, see below. So one can guess that the corollary will follow from the theorem applied to a suitable power of P, to a suitable small set, and to the function w.

More generally, we may apply Theorem IV.2 to the case where there exists a finite number of disjoint small sets, C_k, $k = 1, \dots, s$, associated with the same power of P. Actually, in this case, there exist $m \geq 1$, $b > 0$ and $\nu_k \in \mathcal{P}$, $k = 1, \dots, s$, such that, for any $x \in E$ and $A \in \mathcal{E}$, we have

$$P^m(x, A) \geq b \sum_{k=1}^s 1_{C_k}(x)\nu_k(A) = b\, T_{\nu,\alpha}(x, A),$$

with $\nu = s^{-1} \sum_{k=1}^s \nu_k$ and $\alpha(x, y) = \sum_{k=1}^s 1_{C_k}(x) \frac{d\nu_k}{d\nu}(y)$. For $x \in C_k$, we have $Pw(x) \geq s \int w\, d\nu_k$, so that $\int w d\nu < +\infty$. Since $\alpha^{(w)}(x, y) \leq \sum_{k=1}^s \frac{d\nu_k}{d\nu}(y)w(y)$ which is ν-integrable, we conclude that $\alpha \in \mathcal{H}_\nu^{(w)}$.

For a proper understanding of the hypothesis *(a)* of Theorem IV.2 and later use, we point out the link between this hypothesis and the hitting times of C. From now on, we use the notations

$$\sigma = \sigma_C = \inf\{n : n \geq 0, X_n \in C\}, \quad \tau = \tau_C = \inf\{n : n \geq 1, X_n \in C\}.$$

Lemma IV.1

Let P be a Markov kernel, C be a non empty measurable subset, and let $r_1 > 1$.

(i) Assume there exist a measurable function w from E to $[1, +\infty[$, and a constant $\eta \geq 0$ such that $$Pw \leq r_1^{-1}(w\, 1_{C^c} + \eta\, 1_C).$$

Then $\forall x \in C^c,\ \ E_x[r_1^\sigma] \leq w(x), \qquad \forall x \in C,\ \ E_x[r_1^\tau] \leq \eta.$

(ii) Conversely, assume that there exist a constant η_1 such that

$$\forall x \in E,\ E_x[r_1^\sigma] < +\infty, \quad \forall x \in C,\ E_x[r_1^\tau] \leq \eta_1.$$

Then, setting, for $x \in E$, $w_1(x) = E_x[r_1^\sigma]$, we have

$$\forall x \in E,\ \ Pw_1(x) = r_1^{-1}\big(w_1(x)1_{C^c}(x) + E_x[r_1^\tau]1_C(x)\big) \leq r_1^{-1}\big(w_1(x)\, 1_{C^c}(x) + \eta_1 1_C(x)\big).$$

So the function w_1 appears as being minimal among the functions w verifying *(i)* for given P, C, r_1, and a suitable bound η. If we replace the hypothesis *(a)* of Theorem IV.2 by the hypotheses of the assertion *(ii)* of the preceding lemma, then the conclusions of Theorem IV.2 hold with $w = w_1$. Notice that the inequality $w \geq w_1$ implies that the canonical embedding of B_{w_1} in B_w is continuous.

Proof of Lemma IV.1

(i) Set $M_n = r_1^n w(X_n)$, $n \geq 0$. It follows from the fact that, on C^c, we have $P(r_1 w) \leq w$ that, for $x \in C^c$, $(M_{n\wedge\sigma})_{n\geq 0}$ is a positive P_x-supermartingale. So, for any $n \geq 0$, we get

$$E_x[r_1^{n\wedge\sigma}] \leq E_x[M_{n\wedge\sigma}] \leq E_x[M_0] = w(x).$$

Passing to the limit with respect to n, we get, for $x \in C^c$, $E_x[r_1^\sigma] \leq w(x) < +\infty$.

For $x \in C$, using the above inequality and the fact that $w \geq 1$, we get

$$E_x[r_1^\tau] = r_1 E_x[E_{X_1}[r_1^\sigma]] \leq r_1 P1_C(x) + r_1 P(1_{C^c} w)(x) \leq r_1 Pw(x) \leq \eta.$$

(ii) Let θ be the shift operator. For $x \in E$,

$$Pw_1(x) = E_x[E_{X_1}[r_1^\sigma]] = E_x[E_x[r_1^{\sigma\circ\theta} \mid X_1]] = E_x[r_1^{\sigma\circ\theta}].$$

If $x \in C^c$, $\sigma \circ \theta = \sigma - 1$, while if $x \in C$, $\sigma \circ \theta = \tau - 1$. This gives the equality of the statement, hence the inequality. []

Proof of Theorem IV.2

From *(a)*, we have

$$Pw \leq r_1^{-1} w + r_1^{-1} \eta.$$

This shows that P acts continuously on $\mathcal{B}_w$. Iterating this inequality, we obtain, for any $n \geq 1$, $P^n w \leq r_1^{-n} w + (r_1^{-1}\eta) \frac{1-r_1^{-n}}{1-r_1^{-1}}$. Hence $\sup_n \|P^n\|_w < +\infty$ and $r^w(P) \leq 1$. Since $P1_E = 1_E$, we get $r^w(P) = 1$.

For $x \in E$ and $A \in \mathcal{E}$, we can write

$$P(x, A) = b\, T_{\nu,\alpha}(x, A) + S(x, A),$$

with $S \in \mathcal{T}(E, \mathcal{E})$. Since we assume that the function $\alpha \in \mathcal{H}_\nu^{(w)}$, to apply Theorem IV.1, we have to bound

$$r^w(S) = \lim_n \Big(\sup_{x \in E} \frac{S^n w(x)}{w(x)} \Big)^{1/n}.$$

For this purpose, we use the generating functions G_w^x and G^x, $x \in E$, defined for $r \geq 0$ by

$$G_w^x(r) = \sum_{n \geq 0} r^n \frac{S^n w(x)}{w(x)} \quad \text{and} \quad G^x(r) = \sum_{n \geq 0} r^n S^n(x, E).$$

Lemma IV.2

For $0 \leq r < \min\{1, \|S\|_w^{-1}\}$ the above generating functions converge and we have

$$(1 - \frac{r}{r_1})\, G_w^x(r) \leq 1 + (r_1^{-1}\eta)\, r \frac{G^x(r)}{w(x)}.$$

Proof

The asserted convergences are obvious. We have $Sw \leq Pw \leq r_1^{-1} w + (r_1^{-1}\eta)$. So, for any $n \geq 0$, $S^{n+1} w \leq r_1^{-1} S^n w + (r_1^{-1}\eta) S^n 1_E$. Multiplying the above relation by r^{n+1} and summing for $n \geq 0$, we get

$$w(x)\big(G_w^x(r) - 1\big) \leq r_1^{-1}\, r\, w(x)\, G_w^x(r) + (r_1^{-1}\eta)\, r\, G^x(r).$$

The claimed relation follows. []

So the problem is now to study the generating functions G^x. For this purpose, we show that these generating functions are linked to the behaviour with respect to C of the chain $(Z_n)_{n\geq 0}$ associated with the transition probability P_0. When $\alpha = 1$, P_0 is one of the conditional Markov kernels involved in the definition of the split chain associated with the small set C, [Num]. Notice that, if $b = 1$, the set C is absorbing for $(Z_n)_{n \geq 0}$.

Define the random variables $N_n, n \in \mathbb{N}$, by

$$N_0 = 0 \quad \text{and, for } n \geq 1, \quad N_n = \sharp\{k : k = 0, \ldots, n-1, Z_k \in C\}.$$

For $x \in E$, $k \in \mathbb{N}$, and $r \geq 0$, set

$$L_k^x(r) = \textstyle\sum_{n \geq 0} P_x[N_n = k]\, r^n.$$

Lemma IV.3

For all $x \in E$ and $r \geq 0$, we have in $\mathbb{R}_+ \cup \{+\infty\}$,

$$G^x(r) = \sum_{k\geq 0}(1-b)^k L_k^x(r).$$

Proof

We first establish that, for $x \in E$ and $n \in \mathbb{N}$, $S^n(x,E) = E_x[(1-b)^{N_n}]$.
The formula holds for $n=0$. Proceeding by induction, we assume that it is true at order $n \in \mathbb{N}$. Using the Markov property and the shift operator θ, we get

$$\begin{aligned}
S^{n+1}(x,E) &= \int S(x,dx_1)S^n(x_1,E) = \int P_0(x,dx_1)\big(1-b1_C(x)\big)E_{x_1}[(1-b)^{N_n}] \\
&= E_x\Big[\big(1-b1_C(X_0)\big)E_{X_1}[(1-b)^{N_n}]\Big] = E_x\big[\big(1-b1_C(X_0)\big)\,(1-b)^{N_n\circ\theta}\big] \\
&= E_x\big[(1-b)^{N_{n+1}}\big]\big].
\end{aligned}$$

Since the terms of the two series considered below are positive, we have

$$G^x(r) = \sum_{n\geq 0} r^n \sum_{k\geq 0}(1-b)^k P_x[N_n = k] = \sum_{k\geq 0}(1-b)^k \sum_{n\geq 0} r^n P_x[N_n = k]. \qquad []$$

The sequence $(N_n)_{n\geq 0}$ is similar to a renewall process whose renewall times are the times, ρ_k, $k\geq 0$, of the successive visits of $(Z_n)_{n\geq 0}$ to C. The remainder of the proof is inspired by the method of discrete renewal theory used in the study of Markov chains, cf [Num] Theorem 6.6 or [MeTw] Theorem 15.1.1 (Kendall's Renewall Theorem). To be precise, we set

$$\rho_0 = \inf\{n : n\geq 0, Z_n \in C\}, \qquad \forall k\geq 0,\ \rho_{k+1} = \inf\{n : n > \rho_k, Z_n \in C\},$$

and, for any $x\in E$, $k\geq 0$, and $r\geq 0$, we define the generating functions H_k^x by

$$H_k^x(r) = E_x[r^{\rho_k}].$$

Lemma IV.4

(i) For $x\in E$, $k\geq 0$, and $1\leq r\leq r_1$, we have

$$H_k^x(r) \leq w(x)\,h(r)^k < +\infty,$$

thus the function h and the generating function H_k^x are defined on $D_{r_1} = \{z : z\in \mathbb{C}, |z| < r_1\}$,

(ii) for $k\geq 0$ and $z\in D_{r_1}$, setting $m(z) = \dfrac{r_1\eta_1}{(r_1 - \max\{|z|,1\})^2}$ with $\eta_1 = \max\{r_1, \frac{\eta}{(1-b)}\}$, we have

$$\big|H_k^x(z) - H_{k+1}^x(z)\big| \leq w(x)\,|1-z|\,h(|z|)^k\,m(z),$$

(iii) for $1\leq r < r_1$, $0\leq h(r)-1\leq (r-1)m(r)$.

Proof

(i) If $x\notin C$, $P_0(x,\cdot) = P(x,\cdot)$. It follows that, for $x\notin C$ and $n\in\mathbb{N}$, we have

$$P_x[\rho_0 > n] = P_x\big(\cap_{k=1}^n [Z_k \notin C]\big) = P_x\big(\cap_{k=1}^n [X_k\notin C]\big) = P_x[\sigma > n].$$

So we get, for any $x\notin C$, $H_0^x(r) = E_x[r^\sigma] \leq E_x[r_1^\sigma] \leq w(x)$, by Lemma IV.1.

Since, for $x\in C$, $H_0^x(r) = 1$, we have, for all $x\in E$, $H_0^x(r)\leq w(x)$. Hence *(i)* for $k=0$.

Let us now show that, for $0\leq r\leq r_1$, $h(r) = \sup_{x\in C} E_x[r^{\rho_1}] \leq \eta_1$.

Let $x \in C$. If $b = 1$, we clearly have $E_x[r^{\rho_1}] = r \le r_1$. If $b < 1$, using the preceding and the fact that $P_0(x, \cdot) \le \frac{1}{1-b} P(x, \cdot)$, we have

$$\begin{aligned} E_x[r^{\rho_1}] &= r\Big(P_0 1_C(x) + \int_{C^c} P_0(x, dy) E_y[r^{\rho_0}]\Big) \\ &\le \frac{r}{1-b}\Big(P 1_C(x) + \int_{C^c} P(x, dy) E_y[r^{\sigma}]\Big) = \frac{1}{1-b} E_x[r^{\tau}] \le \frac{\eta}{1-b}. \end{aligned}$$

Hence the claimed bound for h.

Using the strong Markov property, we get, for $k \ge 0$ and $x \in E$,

$$H^x_{k+1}(r) = E_x\big[r^{\rho_k} E_x[r^{(\rho_{k+1}-\rho_k)} | \mathcal{F}_{\rho_k}]\big] = E_x\big[r^{\rho_k} E_{Z_{\rho_k}}[r^{\rho_1}]\big] \le h(r) H^x_k(r),$$

where $\mathcal{F}_{\rho_k}$ is the stopped σ-field. The claimed inequality follows by induction.

(ii) Fix $z \in D_{r_1}$.

From the above bound for h, for all $x \in C$ and all $n \ge 1$, $r_1^n P_x[\rho_1 = n] \le \eta_1$. It follows that

$$(\star)\ \ |1 - H^x_1(z)| \le \sum_{n\ge1} |1 - z^n|\, P_x[\rho_1 = n] \le \frac{\eta_1 |1-z|}{r_1} \sum_{n\ge1} n\Big(\frac{\max\{|z|, 1\}}{r_1^{n-1}}\Big)^{n-1} = |1-z|\, m(z).$$

Assume now $k \ge 1$. We have

$$H^x_k(z) - H^x_{k+1}(z) = E_x[z^{\rho_k}] - E_x\big[z^{\rho_k} E_{Z_{\rho_k}}[z^{\rho_1}]\big] = E_x\big[z^{\rho_k}\big(1 - H_1^{Z_{\rho_k}}(z)\big)\big].$$

so that

$$\big|H^x_k(z) - H^x_{k+1}(z)\big| \le E_x\big[|z|^{\rho_k}\big] \sup_{x\in C} \big|1 - H^x_1(z)\big|.$$

Using *(i)* and $(\star)$, we get the stated inequality.

For $1 \le r \le r_1$, $(\star)$ gives $0 \le H^x_1(r) - 1 \le (r-1)m(r)$. *(iii)* follows by getting to the supremum for $x \in C$. []

End of the proof of Theorem IV.2

For $k \in \mathbb{N}$ and $z \in D_{r_1} \backslash \{1\}$, set

$$R^x_k(z) = \frac{H^x_k(z) - H^x_{k+1}(z)}{1-z}.$$

This function is holomorphic in $D_{r_1} \backslash \{1\}$, and, from *(ii)* of Lemma IV.4, it verifies

$$|R^x_k(z)| \le w(x)\, h(|z|)^k m(z).$$

This implies that it is bounded in a neighbourhood of 1, and thus it extends to an holomorphic function on D_{r_1}, which is still denoted by R^x_k.

Since $[N_n = k] = [\rho_k \le n] \backslash [\rho_{k+1} \le n]$, for $z \in \mathbb{C}$, $|z| < 1$, we have

$$\begin{aligned} L^x_k(z) &= \sum_{n\ge0} z^n P_x[\rho_k \le n] - \sum_{n\ge0} z^n P_x[\rho_{k+1} \le n] \\ &= \frac{H^x_k(z) - H^x_{k+1}(z)}{1-z} = R^x_k(z). \end{aligned}$$

Thus the generating function L_k^x is the Taylor expansion of R_k^x at 0, consequently it converges on D_{r_1}, and, for any $z \in D_{r_1}$, we have

$$L_k^x(z) = R_k^x(z).$$

Returning to the formula of Lemma IV.3, we get, for $0 \leq r < r_1$,

$$G^x(r) = \sum_{k \geq 0} (1-b)^k R_k^x(r) \leq m(r) \sum_{k \geq 0} [(1-b)h(r)]^k \, w(x).$$

By *(iii)* in Lemma IV.4, $\lim_{r \to 1+} h(r) = 1$, so, since $b > 0$, there exists an $r > 1$ such that $(1-b)h(r) < 1$ and the number r_b defined in *(i)* of the statement is > 1.

Suppose r such that $1 \leq r < r_b$. We have

$$\sup_{x \in E} \frac{G^x(r)}{w(x)} \leq \frac{m(r)}{1-(1-b)h(r)} = M(r) < +\infty.$$

From Lemma IV.2, we deduce that the series $G_w^x(r)$ converges and that

$$G_w^x(r) \leq \frac{r_1}{r_1 - r}\Big(1 + (r_1^{-1}\eta) r M(r)\Big) = M_w(r).$$

It follows that, for any $n \geq 0$, $r^n \sup_{x \in E} \frac{S^n w(x)}{w(x)} \leq M_w(r)$, this shows that $r^w(S) \leq r^{-1}$. Finally $r^w(S) \leq 1/r_b < 1$.

To complete the proof it suffices now to apply Theorem IV.1. []

Proof of Corollary IV.2

It follows from Lemma 15.2.2 in [MeTw], that E_1 is absorbing (it is also full, i.e. $\psi(E_1^c) = 0$, for any maximal irreducibility measure ψ). From now on, we assume $E_1 = E$.

Now we prove that the hypotheses *(a)* and *(b)* of Theorem IV.2 hold, this will yield quasi-compactness on $\mathcal{B}_w$. Notice that, if we suppose that $\sup_C w = \beta < +\infty$, we get from *(a')*

$$Pw \leq \rho w + \zeta 1_C \leq \rho w 1_{C^c} + (\rho\beta + \zeta) 1_C,$$

so that *(a)* and *(b)* are verified for P, w, and C. To treat the general case, we introduce the level subsets of w, $C_t = \{x : x \in E, w(x) \leq t\}$, $t \geq 1$, and we prove that *(a)* and *(b)* are fulfilled for w, a suitable C_t and a certain power P^m.

The relation in *(a')* implies $Pw \leq \rho w + \zeta$. Iterating this inequality, we get, for any $n \geq 1$,

$$P^n w \leq \rho^n w + \zeta \tfrac{1-\rho^n}{1-\rho} \leq \rho \, w + \tfrac{\zeta}{1-\rho}.$$

Choose t_0 such that $\overline{\rho} = \rho + \frac{\zeta}{(1-\rho)t_0} < 1$. For $t \geq t_0$ and any $n \geq 1$, setting $\zeta_t = \rho \, t + \frac{\zeta}{1-\rho}$, we have

$$P^n w \leq (\rho \, w + \tfrac{\zeta}{1-\rho} \, \tfrac{w}{t}) 1_{C_t^c} + (\rho \, t + \tfrac{\zeta}{1-\rho}) 1_{C_t} \leq \overline{\rho} \, w \, 1_{C_t^c} + \zeta_t 1_{C_t}.$$

Following [MeTw] Section 11.3.2, we now prove that, for $t \geq t_0$, C_t is a small set. As already pointed out, in the present context, “petite” sets are identical to small sets. Fix m, $b > 0$ and $\nu \in \mathcal{P}$ such that

$$P^m \geq b 1_C \nu.$$

Since, for $x \in C^c$, $Pw(x) \leq \rho w(x)$ and $w \geq 1$, by Lemma IV.1, we have, for any $x \in E$, $E_x[\rho^{-\sigma_C}] \leq w(x)$. Using the Markov inequality, we get, for $x \in C_t$ an $k \geq 0$,

$$\rho^{-(k+1)} P_x[\sigma_C \geq k+1] \leq w(x) \leq t.$$

So there exists k_t such that, for $x \in C_t$,

$$P_x\Big(\cup_{k=0}^{k_t}[X_k \in C]\Big) = P_x[\sigma_C \le k_t] \ge 1/2.$$

Consequently, for any $x \in C_t$, there exist k, $0 \le k \le k_t$, depending on x, such that $P_x[X_k \in C] \ge \frac{1}{2(k_t+1)}$. For any $A \in \mathcal{E}$, we have

$$P^{k+m}(x,A) \ge \int_C P^k(x,dy)P^m(y,A) \ge \tfrac{b}{2(k_t+1)}\nu(A).$$

It follows that C_t is a "petite set", hence a small set associated with some power P^{n_t} of P. Since $t \ge t_0$, we have $P^{n_t} \le \overline{\rho}\, w\, 1_{C_t^c} + \zeta_t 1_{C_t}$.

The facts that 1 is the only modulus 1 eigenvalue and that it is simple follow from irreducibility and aperiodicity, and this implies geometric ergodicity, see Corollary IV.3 below. []

IV.3 Converse and ergodic theorem

The assertion of quasi-compactness of Theorem IV.2 has a converse that we now establish.

Theorem IV.3

Assume the Markov kernel P is quasi-compact on $\mathcal{B}_w$. Set, for $t \ge 1$, $C_t = \{x : x \in E, w(x) \le t\}$. Then, for any $t \ge 1$ and $0 < b < 1$, there exist n_t, $\rho < 1$, and $\eta \ge 0$, such that, for any $n \ge n_t$,

(i) $P^n w \le \eta\, 1_{C_t} + \rho^n w\, 1_{C_t^c}$,

(ii) there exist $\nu \in \mathcal{P}$ and $\alpha \in \mathcal{H}_\nu^{(w)}$ such that $T_{\nu,\alpha}$ is Markov and $P^n \ge b\, 1_{C_t} T_{\nu,\alpha}$.

Proof

The proof is based on the use of Theorem III.2. The fact that P is Markov allows to specify the structure of the kernel L in point *(i)* of this theorem.

Lemma IV.5

There exist bounded kernels L and N and $\rho_1 < 1$ such that

$$P = L + N, \quad LN = NL = 0, \quad r(N) < \rho_1,$$
$$\forall (x,A) \in E \times \mathcal{E}, \quad L(x,A) = \textstyle\sum_{k=1}^{s} \lambda_k f_k(x)\varphi_k(A),$$

where, for $k = 1, \dots, s$, $|\lambda_k| = 1$, $f_k \in \mathcal{B}$, $\varphi_k \in \mathcal{M}$, and $v(\varphi_k)(w) < +\infty$.

Proof of Lemma IV.5

Applying the point *(i)* of Theorem III.2 to $P^{(w)}$ we see that there exist kernels L and N acting on $\mathcal{B}_w$ verifying the required properties, except that L and N may be unbounded, that $r^w(P)$ is unknown, and that we can only assert that L has a finite dimensional range and that its non zero eigenvalues have modulus $r^w(P)$.

To L we can associate functions $f_i \in F = L(\mathcal{B}_w)$, functionals $\varphi_i \in \mathcal{B}_w{}'$, $i = 1, \dots, s$, verifying $\varphi_j(f_i) = \delta_{ij}$, and a complex triangular invertible matrix $M = [m_{ij}]_{i,j=1}^{s}$ such that, setting $M^n = [m_{ij}^{(n)}]$, we have, for any $n \ge 1$ and $f \in \mathcal{B}_w$,

$$L^n f = \sum_{i=1}^{s}\Big(\sum_{j=1}^{s} m_{ij}^{(n)}\varphi_j(f)\Big) f_i.$$

As in the proof of point *(ii)* of Theorem III.2, it is possible to construct finitely supported measures μ_k, $k = 1, \dots, s$, such that for $k, m = 1, \dots, s$, $\int f_m d\mu_k = \delta_{m,k}$. So,

for $i = 1, \dots, s$, we have

$$\mu_i(L^n f) = \sum_{j=1}^{s} m_{ij}^{(n)} \varphi_j(f).$$

Taking $n = 1$, it follows from the facts that M is invertible and that L is a kernel acting on $\mathcal{B}_w$ that φ_j is a non zero measure on E such that $\int w \, dv(\varphi_j) < +\infty$.

Consider the linear application V from $\mathcal{B}$ to $\mathbb{C}^n$ defined by $V(f) = \big(\varphi_i(f)\big)_{i=1}^{s}$. Suppose $V(\mathcal{B}) \neq \mathbb{C}^s$, then there exists $(a_j)_{j=1}^{s} \in \mathbb{C}^s \backslash \{0\}$ such that, for any $f \in \mathcal{B}$, we have $\sum_{i=1}^{s} a_i \varphi_i(f) = 0$, i.e $\sum_{i=1}^{s} a_i \varphi_i$ is zero on $\mathcal{B}$. Since $\sum_{i=1}^{s} a_i \varphi_i$ is a measure, it follows from Lebesgue's Theorem, that $\sum_{i=1}^{s} a_i \varphi_i = 0$, but, by assumption the functionals φ_i, $i = 1, \dots, s$, are linearly independant. So V is onto. It follows that there exists $g_j \in \mathcal{B}$, such that, for any $i, j = 1, \dots, s$, $\varphi_j(g_i) = \delta_{ij}$. Consequently, for any $i, j = 1, \dots, s$ and n, we have

$$m_{ij}^{(n)} = \mu_i(L^n g_j) = \mu_i(P^n g_j) - \mu_i(N^n g_j).$$

We choose ρ_1 verifying $r^w(N) < \rho_1 < r^w(P)$, and then n_1 such that, for any $n \geq n_1$, $\|N^n\|_w \leq \rho_1^n$, i.e, for any $f \in \mathcal{B}_w$ and all $x \in E$, we have $|N^n f(x)| \leq \rho_1^n \|f\|_w w(x)$. Since g_j is bounded and P is Markov, $\big(\mu_i(P^n g_j)\big)_{n \geq 1}$ is bounded. μ_i is finitely supported so it integrates w, hence $\big(\rho_1^{-n} \mu_i(N^n g_j)\big)_{n \geq 1}$ is bounded. It follows that there exists a constant C such that, for any $n \geq 1, i, j = 1, \dots, s$,

$$(\star\star) \qquad |m_{ij}^{(n)}| \leq C(1 + \rho_1^n).$$

In particular, $r^w(P)^n = |m_{11}^{(n)}| \leq C(1 + \rho_1^n)$. But, since $\rho_1 < r^w(P)$, this implies $r^w(P) \leq 1$. Using $P1_E = 1_E$, we conclude that $r^w(P) = 1$.

The eigenvalues of the triangular matrix M have modulus 1, and, by $(\star\star)$, the powers M^n, $n \geq 1$ are bounded, so M is in fact diagonal. Setting $M = \mathrm{diag}(\lambda_1, \dots, \lambda_s)$, with $|\lambda_i| = 1$, $i = 1, \dots, s$, we get, for any $n \geq 1$ and $f \in \mathcal{B}_w$,

$$L^n f = \textstyle\sum_{i=1}^{s} \lambda_i^n \varphi_i(f) f_i.$$

From the relation $L^n g_j = \lambda_j^n f_j$, using $\rho_1 < 1$, we deduce that, for any $x \in E$,

$$|f_j(x)| = \lim_n |L^n g_j(x)| \leq \limsup_n |P^n g_j(x)| + \limsup_n |N^n g_j(x)| \leq \|g_j\|.$$

Thus f_j is a bounded function. []

Proof of (i) With the notations of the lemma, for any $n \geq 1$ and $x \in E$, we have

$$L^n w(x) \leq \textstyle\sum_{i=1}^{s} |\varphi_i(w)| \, \|f_i\| = \zeta.$$

Otherwise, there exists n_1 such that, for any $n \geq n_1$, $|N^n w(x)| \leq \rho_1^n w(x)$. Consequently, for $n \geq n_1$,

$$P^n w \leq \zeta + \rho_1^n w.$$

Introducing the set C_t, the above inequality leads to

$$P^n w \leq (\zeta + \rho_1^n t) 1_{C_t} + (\zeta + \rho_1^n w) 1_{C_t^c}.$$

Let ρ, $\rho_1 < \rho < 1$. For $x \in C_t^c$, we have $\rho^n w(x) - (\zeta + \rho_1^n w(x)) \geq (\rho^n - \rho_1^n)t - \zeta$, we choose $n_2 \geq n_1$ such that, for $n \geq n_2$, this last number is ≥ 0. Setting $\eta = \zeta + t$, we get, for $n \geq n_2$, $P^n w \leq \eta 1_{C_t} + \rho^n w \, 1_{C_t^c}$.

Proof of (ii) Let ρ as above and n_3 such that, for $n \geq n_3$, $t\rho^n \leq 1 - b$. Let $n_4 = \max\{n_3, \ell_\rho\}$, where ℓ_ρ is defined in point *(iii)* of Theorem III.2 applied to $P^{(w)}$. For any $n \geq n_4$, we see that there exist $\nu \in \mathcal{P}$ and $\alpha' \in \mathcal{H}_\nu^{(w)}$, such that $S = P^n - T_{\nu,\alpha'} \geq 0$, and

$\|S\|_w < \rho^n$. In particular with the function $f = 1_E$, we get, for $x \in C_t$, $0 \le 1 - T_{\nu,\alpha'} 1_E(x) \le t\rho^n$. So on C_t, $T_{\nu,\alpha'} 1_E \ge b$. It follows that, for $x \in C_t$ and $A \in \mathcal{E}$, we have

$$P^n(x, A) \ge T_{\nu,\alpha'}(x, A) \ge b \frac{T_{\nu,\alpha'}(x, A)}{T_{\nu,\alpha'}(x, E)} = b 1_{C_t}(x) T_{\nu,\alpha}(x, A),$$

with $\alpha(x, y) = T_{\nu,\alpha'}(x, E)^{-1} \alpha'(x, y)$. Since $\alpha(x, y) \le b^{-1} \alpha'(x, y)$, we see that $\alpha \in \mathcal{H}_\nu^{(w)}$. []

Corollary IV.3 *Ergodic Theorem*

Assume that the Markov kernel P is quasi-compact on $\mathcal{B}_w$.

Then there exist an integer $d \ge 1$, a finite rank Markov kernel S such that $S(\mathcal{B}_w) \subset \mathcal{B}$, and constants C and $0 \le \rho < 1$, such that, for any $k = 0, \ldots, d-1$, and $n \ge 0$

$$\|P^{nd+k} - P^k S\|_w \le C\rho^n,$$

that is, for any $f \in \mathcal{B}_w$ and $x \in E$,

$$|P^{nd+k} f(x) - P^k S f(x)| \le C \|f\|_w \rho^n w(x).$$

If moreover P is irreducible and aperiodic, then $d = 1$ and S is defined by $Sf = \pi(f) 1_E$, where π is the unique P-invariant probability distribution.

This statement is similar to a part of the result known for Markov kernels having a quasicompact action on the space $\mathcal{B} = \mathcal{B}_1$ of bounded measurable functions [Nev1], [Rev]. In fact our theorem has to be completed by the description of the precise structures of the Markov kernels $P^k S$, $k = 0, \ldots, d-1$. But it is easily seen that, for this purpose, the arguments developped in the case of a quasi-compact action on $\mathcal{B}$ apply to the present context. The same representation in terms of invariant probability measures, absorbing sets, and aperiodic classes can be obtained. See, for example, the two last paragraphs of the proof of Theo. 3.7, Chap. 6, [Rev]. It follows that, under the irreducibility and aperiodicity hypotheses, the special assertion in the corollary is a consequence of the general one.

Notice that, as a consequence of Lemma IV.5, there exists a finite rank Markov kernel P_1 such that $P_1(\mathcal{B}_w) \subset \mathcal{B}$, and, for any $f \in \mathcal{B}_w$,

$$\lim_n \left\| \frac{1}{n} \sum_{k=1}^{n-1} P^k f - P_1 f \right\|_w = 0.$$

The above ergodic theorem has been established in [Wu] on the basis of a general statement for positive quasi-compact operators on Banach lattices. Here, we shall sketch a proof based on the use of space-time harmonic functions, adapting the method described in [Rev] Chap 6, Section 3.

Proof

Let $\mathcal{H}_w$ be the space of w-bounded complex valued space-time harmonic functions for P, that is the set of sequences $(h_n)_{n \ge 0}$ of elements of $\mathcal{B}_w$ such that $\sup_{n \ge 0} \|h_n\|_w < +\infty$, and, for any $n \ge 0$, $P h_{n+1} = h_n$. A straightforward adaptation of [Rev] Prop. 3.6 leads to

Lemma IV.6

$\mathcal{H}_w$ is finite dimensional.

Let $\mathcal{H}$ be the space of space-time harmonic bounded functions. Since $\dim \mathcal{H} \le \dim \mathcal{H}_w < +\infty$, the structure of elements of $\mathcal{H}$ is described by [Rev] Prop. 3.5. In particular, there exists an integer $d \ge 1$ such that, for $(h_n)_{n \ge 0} \in \mathcal{H}$, we have, for any $n \ge 0$, $h_{n+d} = h_n$. The sequences $(\lambda_i^{-n} f_i)_{n \ge 0}$, $i = 1, \ldots, s$, where λ_i is a modulus 1 eigenvalue of

P and f_i a corresponding eigenfunction, are in $\mathcal{H}$ as shown by Lemma IV.5, so, we have $\lambda_i^d = 1$. For any $n \geq 1$, $P^{nd} = L^d + N^{nd}$, with $r^w(N) < 1$, setting $S = L^d$, the claimed convergences follow from Lemma IV.5. []

Remark *Central Limit Theorems*

Let $(X_n)_{n\geq 0}$ be a Markov chain on $(E, \mathcal{E})$ associated with a Markov kernel P which has a quasi-compact action on $\mathcal{B}_w$ for a suitable w, and let ξ be a measurable real valued function on E. As mentioned in the introduction, quasi-compactness of P is a usefull tool to get central limit theorems for the sequence of random variables $\big(\xi(X_n)\big)_{n\geq 0}$, see [HenHer] for a general description of the method. In the case of geometric ergodicity on $\mathcal{B}_w$, using a refinement of this method and Corollary III.2, L. Hervé [Her] has established limit theorems for functions ξ which are dominated by a suitable power w^α. When $\alpha < 1/4$ the Central Limit Theorem with a rate of convergence holds, while $\alpha < 1/2$ is sufficient for the Local Theorem.

V. APPENDIX

V.1 Proofs of the results of Section II

In this subsection, $\mathcal{B}$ is an abstract Banach space.

Proof of Theorem II.1

Set $u_n = \inf\{\|Q^n - V\| : V \in \mathcal{K}(\mathcal{B})\}$. It is easy verified that $(u_n)_{n\geq 1}$ is submultiplicative, so that the sequence $(u_n^{1/n})_{n\geq 1}$ has a limit denoted $r_K(Q)$,

$$r_K(Q) = \lim_n \big(\inf\{\|Q^n - V\| : V \in \mathcal{K}(\mathcal{B})\}\big)^{1/n}.$$

The statement of Theorem II.1 is now $r_e(Q) = r_K(Q)$.

A. $r_K(Q) \leq r_e(Q)$

Clearly $r_K(Q) \leq r(Q)$, so that the inequality is proved if $r_e(Q) = r(Q)$. Suppose $r_e(Q) < \rho \leq r(Q)$. Using the direct sum decomposition of Definition II.1, we have, for any $n \geq 1$, $Q^n = L^n + N^n$, where L has a finite rang and $r(N) < \rho$. It follows that $r_K(Q) \leq r(N) < \rho$. Hence $r_K(Q) \leq r_e(Q)$.

B. $r_K(Q) \geq r_e(Q)$

It is convenient here to introduce a function defined on the set of bounded sequences on $\mathcal{B}$.

Definition V.1

For a bounded sequence $(h_n)_{n\geq 1}$ in $\mathcal{B}$, we set

$$\gamma_\sigma\big((h_n)_{n\geq 1}\big) = \inf_{N\geq 1} \sup_{n,n'\geq N} \|h_n - h_{n'}\|.$$

The equality $\gamma_\sigma\big((h_n)_{n\geq 1}\big) = 0$ means that $(h_n)_{n\geq 1}$ is a Cauchy sequence. The number $\gamma_\sigma\big((h_n)_{n\geq 1}\big)$ may be seen to measure how much the bounded sequence $(h_n)_{n\geq 1}$ differs from a Cauchy sequence. It may be compared to the set function γ introduced by R. D. Nussbaum [Nus] which associate to a bounded subset C of $\mathcal{B}$, the number $\gamma(C)$ equal to the infimum of the collection of positive real numbers r for which there exists a finite covering of C by open balls of radius r. The key property is the following.

Lemma V.1

Let $\rho > r_K(Q)$. Then there exists a constant C such that, for any bounded sequence $(f_n)_{n\geq 1}$ in $\mathcal{B}$, there exists a subsequence $(n_k)_{k\geq 1}$ such that, for any $s \geq 1$,

$$\gamma_\sigma\big((Q^s f_{n_k})_{k\geq 1}\big) \leq C\rho^s\, \gamma_\sigma\big((f_{n_k})_{k\geq 1}\big) \leq 2C\rho^s \sup_{k\geq 1} \|f_{n_k}\|.$$

Proof

It follows from the definition of $r_K(Q)$ that there exists a sequence $(V_s)_{s\geq 1}$ of compact operators of $\mathcal{B}$ and a real number C such that, for any $s \geq 1$, $\|Q^s - V_s\| \leq C\rho^s$. Since, for any $s \geq 1$, the sequence $(V_s f_n)_{n\geq 1}$ is conditionally compact, using the Cantor diagonal process, we can construct a subsequence $(n_k)_{k\geq 1}$ such that, for any $s \geq 1$, the sequence $(V_s f_{n_k})_{k\geq 1}$ converges. Consequently, for any $s \geq 1$, $\gamma_\sigma\big((V_s f_{n_k})_{k\geq 1}\big) = 0$. The statement then follows from the inequality

$$\|Q^s f_{n_k} - Q^s f_{n_{k'}}\| \leq \|V_s f_{n_k} - V_s f_{n_{k'}}\| + C\rho^k \|f_{n_k} - f_{n_{k'}}\|. \qquad []$$

The above lemma has two corollaries calling back to the theory of compact operators.

Lemma V.2

Let z, $|z| > r_K(Q)$, $g, g_n, f_n \in \mathcal{B}$ such that

$$g_n = (z - Q)f_n,\ \sup_n \|f_n\| < +\infty,\ \lim_n \|g_n - g\| = 0.$$

Then there exist $(n_k)_k$ and $f \in \mathcal{B}$ such that

$$\lim_k \|f_{n_k} - f\| = 0, \quad zf - Qf = g.$$

Proof

Using the relation $z^s - Q^s = Q_s(z - Q)$, where $Q_s = \sum_{\ell=0}^{s-1} z^\ell Q^{s-1-\ell}$, we write

$$z^s f_n = Q^s f_n + Q_s g_n.$$

Applying Lemma V.1 with $\rho = \frac{|z| + r_K(Q)}{2}$ and noticing that $(Q_s\, g_{n_k})_{k\geq 1}$ converges, we get

$$|z^s|\gamma\big((f_{n_k})_{k\geq 1}\big) \leq 2C\rho^s \sup_{k\geq 1} \|f_{n_k}\| + \gamma\big((Q_s g_{n_k})_{k\geq 1}\big) = 2C\rho^s \sup_{k\geq 1} \|f_{n_k}\|.$$

Dividing by $|z|^s$ and letting s tend to infinity, we obtain $\gamma\big((f_{n_k})_{k\geq 1}\big) = 0$. Hence the convergence of $(f_{n_k})_{k\geq 1}$ and the claimed assertion. []

Lemma V.3

Let $\rho > r_K(Q)$, let $J \subset \mathbb{Z}$ such that, if m et $n \in J$, $[m, n] \cap \mathbb{Z} \subset J$, and let z_n, $n \in J$, be a bounded sequence in $C_\rho = \{z : z \in \mathbb{C}, |z| \geq \rho\}$, let F_n, $n \in J$, be closed subspaces of $\mathcal{B}$, such that, if $n, n+1 \in J$, $F_n \subset F_{n+1}, F_n \neq F_{n+1}$.

Assume that, for any $n \in J$, we have

$$Q(F_n) \subset F_n \quad \textit{and, if} \quad n - 1 \in J,\ (z_n - Q)(F_n) \subset F_{n-1},$$

Then J is finite.

Proof

Suppose J is not finite.

From Riesz Lemma [DS] VII-4-3, if $n, n+1 \in J$, there exists $f_{n+1} \in F_{n+1}$ such that

$$\|f_{n+1}\| = 1 \quad \text{et} \quad d(f_{n+1}, F_n) = \inf\{\|f_{n+1} - h\| : h \in F_n\} \geq 1/2.$$

Let z be a limit value of $(z_n)_n$. Set $\rho' = \frac{\rho + r_K(Q)}{2}$ and choose the integer s such that $2C\big(\frac{\rho'}{|z|}\big)^s < 1/8$, and set $h_n = z^{-s}Q^s f_n$. From Lemma V.1, applied with ρ', we get

$$\gamma_\sigma\big((h_n)_{n\geq 1}\big) \leq |z|^{-s}\, 2C\rho'^s \sup_{n\geq 1} \|f_n\| < 1/8.$$

For $n, n+p \in J,\ p > 0$, we have

$$h_{n+p} - h_n = f_{n+p} - \widetilde{f}_{n,p} + (z^{-s} - z_{n+p}^{-s})Q^s f_{n+p},$$

where

$$\begin{aligned}\widetilde{f}_{n,p} =& [f_{n+p} - z_{n+p}^{-s} Q^s f_{n+p}] + z^{-s}Q^s f_n \\ =& [\sum_{i=0}^{s-1} z_{n+p}^{-i} Q^i](f_{n+p} - z_{n+p}^{-1} Q f_{n+p}) + z^{-s}Q^s f_n \quad \in F_{n+p-1},\end{aligned}$$

hence $\quad \|h_{n+p} - h_n\| \geq \|f_{n+p} - \widetilde{f}_{n,p}\| - |z^{-s} - z_{n+p}^{-s}|\|Q^s\| \geq 1/2 - |z^{-s} - z_{n+p}^{-s}|\|Q^s\|.$

Since z is a limit value of $(z_n)_n$, there exists a subsequence $(n_k)_k$ such that, for all k and ℓ, $\|h_{n_{k+\ell}} - h_{n_k}\| \geq 1/4$. So $\gamma_\sigma\big((h_n)_{n\geq 1}\big) \geq 1/4$, in contradiction with what was previously stated. So we conclude that J is finite. []

Now to proceed with the proof of Theorem II.1, one has only to adapt the standard arguments of the theory of compact operators. More precisely the remainder of the proof is contained in the Lemmas XIV-7, 8, 9 of [HenHer], where however the real number r has to be replaced by $r_K(Q)$. []

Proof of Corollary II.1

(i) follows straightforwardly from the formula of Theorem II.1.

(ii) Let $\rho > r_e(Q)$. Since $r_e(Q) = r_K(Q)$, the key property of Lemma V.1 holds for Q and so it also holds for the restriction $Q_{|G}$. It then follows from the part **B** of the proof of Theorem II.1 that $\rho > r_e(Q_{|G})$. We conclude that $r_e(Q) \geq r_e(Q_{|G})$.

(iii) Denote by $\mathcal{K}(\mathcal{B})$ (resp. $\mathcal{K}(\mathcal{B}')$) the ideal of compact operators on $\mathcal{B}$ (resp. $\mathcal{B}'$). It is known, [DS], VI-5-2, that $\big(\mathcal{K}(\mathcal{B})\big)' \subset \mathcal{K}(\mathcal{B}')$. It follows that, for each $n \geq 1$,

$$\inf\big\{\|Q'^n - W\| : W \in \mathcal{K}(\mathcal{B}')\big\} \leq \inf\big\{\|Q'^n - V'\| : V \in \mathcal{K}(\mathcal{B})\big\} = \inf\big\{\|Q^n - V\| : V \in \mathcal{K}(\mathcal{B})\big\}.$$

Hence $r_e(Q') \leq r_e(Q)$.

Let $\mathcal{B}''$ be the topological dual space of $\mathcal{B}'$ and Q'' be the adjoint of Q'. We have $r_e(Q'') \leq r_e(Q')$. Denote by J the canonical embedding of $\mathcal{B}$ into $\mathcal{B}''$. J is an isometry and, since $\mathcal{B}$ is a Banach space, $J(\mathcal{B})$ is closed in $\mathcal{B}''$. As ${Q''}_{|J(\mathcal{B})} = JQJ^{-1}$, we have $r_e({Q''}_{|J(\mathcal{B})}) = r_e(Q)$. Using *(ii)* and the two previous relations, we get $r_e(Q) \leq r_e(Q')$. []

V.2 Lebesgue-Nikodym's decomposition of kernels

Lemma V.4

Suppose that $\mathcal{E}$ is countably generated. Let $R \in \mathcal{T}(E, \mathcal{E})$ and K be a bounded kernel. Then there exist a measurable function β from $E \times E$ to $\mathbb{C}$ and a kernel S such that

(i) $sup_{x\in E} \int |\beta(x,y)| R(x, dy) < +\infty$,

(ii) for any $x \in E$, $S(x,\cdot)$ is singular with respect to $R(x,\cdot)$,
(iii) for any $(x,A) \in E \times \mathcal{E}$, $K(x,A) = \int_A \beta(x,y)R(x,dy) + S(x,A)$,
(iv) moreover, if there exist a measurable function b on E such that, for any $(x,A) \in E\times\mathcal{E}$, $|K(x,A)| \leq b(x)R(x,A)$ then $|\beta| \leq b$ and $S = 0$.

This result is well known, cf, for example, [Num], when the kernel R doesn't depend on the variable x. We adapt below the proof of this standart case.

Proof

For any $n \geq 1$, let $\Pi_n = \{B_k^n : k = 1, \ldots, p_n\}$ be a partition of E by elements of $\mathcal{E}$, such that Π_{n+1} is a refinement of Π_n, and $\mathcal{E}$ is generated by $\cup_{n\geq 1}\Pi_n$.

Let $x \in E$ such that $R(x,E) > 0$. Then $R_1(x,\cdot) = R(x,E)^{-1}R(x,\cdot)$ is a probability distribution. The Lebesgue-Nikodym' decomposition of the finite complex measure $K(x,\cdot)$ on $\mathcal{E}$ with respect to $R_1(x,\cdot)$ can be written

$$A \in \mathcal{E}, \quad K(x,A) = \int_A \varphi(x,y)R_1(x,dy) + K(x, A \cap N^x),$$

where $\varphi(x,\cdot)$ is $R_1(x,\cdot)$-integrable and $R_1(x,N^x) = 0$. Consider the restriction of the probability $R_1(x,\cdot)$ and of the measure $K(x,\cdot)$ to the σ-field $\sigma(\Pi_n)$ generated by the partition Π_n. The absolutely continuous part of the Lebesgue-Nikodym' decomposition of this restriction can be defined by the density

$$\varphi_n(x,y) = \sum_{k:k=1,\ldots,p_n, R_1(x,B_k^n)>0} \frac{K(x,B_k^n)}{R_1(x,B_k^n)} 1_{B_k^n}(y).$$

Using the Jordan' decompositions of $\Re K(x,\cdot)$ and $\Im K(x,\cdot)$, it is seen that $(\varphi_n(x,\cdot))_{n\geq 1}$ is a combination with coefficient $+1$, -1, i, $-i$ of four positive supermartingales with respect to the filtration $\big(\sigma(\Pi_n)\big)_{n\geq 1}$ on the probability space $(E, \mathcal{E}, R_1(x,\cdot))$. Moreover there exist $E_x \in \mathcal{E}$ such that $R_1(x,E_x) = 1$ and, for any $y \in E_x$, $\lim_n \varphi_n(x,y) = \varphi(x,y)$. See for example [Nev2] Prop. III.2.7, for a proof of the above assertions. The functions φ_n, $n \geq 1$, are clearly $\mathcal{E} \times \mathcal{E}$-measurable. Setting, for any $x, y \in E$, $\beta_1(x,y) = 1_{\{x:R_1(x,E)>0\}}(x) \liminf \varphi_n(x,y)$, we get a $\mathcal{E} \times \mathcal{E}$-measurable function such that, for any $x \in E$, $\beta_1(x,\cdot) = \varphi(x,\cdot)$, $R_1(x,\cdot)$ almost surely. To conclude we define β on $E \times E$ by $\beta(x,y) = R(x,E)\beta_1(x,y)$.

The assertion concerning boundedness follows from the definitions of φ_n and β. []

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